A COMPREHENSIVE REVIEW ON A CLASS OF PLANAR WELL- COVERED GRAPHS WITH GIRTH FOUR

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Abstract:

This proposed paper focuses on Comprehensive Review on a class of planar well-covered graphs with girth four. The various definitions, theorems, lemma, corollary and properties are studied in class of planar well-covered graphs with girth four. The detailed studies has been done in a class of planar well-covered graphs with girth four, well-covered graphs and extendability, very well covered graphs and strongly well-covered graphs. The proposed studies in this paper work have imparted the significance of a study on a class of planar well-covered graphs with girth four in graph theory.

Keywords: Class of Planar Well-Covered Graphs with Girth Four, Well-Covered Graphs & Graph Theory

1. Introduction:

In this proposal basic definitions and terminologies are found. A Set of points in a graph is independent if no two points in the graph are joined by a line. The maximum size possible for a set of independent points in a graph G is called the independence number of G and is denoted by α (G)[1]. A set of independent points which attains the maximum size is referred to as a maximum independent set. A set S of independent points in a graph is maximal with respect to set inclusion if the addition to S of any other point in the graph destroys the independence.

In general, a maximal independent set in a graph is not necessarily maximum. Introduced the notion of considering graphs in which every maximal independent set is also maximum. He called a graph having this property a well-covered graph [2]. The work on well-covered graphs that has appeared in the literature has focused on certain subclasses of well-covered graphs. Characterized all cubic well-covered graphs with connectivity at most two, and proved that there are only four 3-connected cubic planar well-covered graphs have recently completed the picture for cubic well covered graphs by determining all 3-connected cubic well-covered graphs.

For a well-covered graph with no isolated points, the independence number is at most one-half the size of the graph. Well-covered graphs whose independence number is exactly one-half the size of the graph is called very well-covered graphs [3]. The subclass of very well-covered graphs was characterized by includes all well-covered trees and all well- covered bipartite graphs. Independently characterized bipartite well-covered graphs and characterized the very well-covered graphs. Recently characterized the very well-covered graphs as a subset of a more general than well-covered class of graphs [4].

1.1 Basic Definitions:

- ✓ In graph theory, a planar graph is a graph that can be embedded in the plane, i.e., it can be drawn on the plane in such a way that its edges intersect only at their endpoints. In other words, it can be drawn in such a way that no edges cross each other.

- ✓ In graph theory, the girth of a graph is the length of a shortest cycle contained in the graph. If the graph does not contain any cycles (i.e. it's an acyclic graph), its girth is defined to be infinity.
- ✓ In graph theory, a well-covered graph is an undirected graph in which every minimal vertex cover has the same size as every other minimal vertex cover. Equivalently, these are the graphs in which every maximal independent set has the same size.
- ✓ A bipartite graph G is well-covered if and only if it has a perfect matching M with the property that, for every edge uv in M, the induced subgraph of the neighbors of u and v forms a complete bipartite graph.
- ✓ An independent set in a graph is a set of vertices no two of which are adjacent to each other. If C is a vertex cover in a graph G, the complement of C must be an independent set, and vice versa. C is a minimal vertex cover if and only if its complement I is a maximal independent set, and C is a minimum vertex cover if and only if its complement is a maximum independent set. Therefore, a well-covered graph is, equivalently, a graph in which every maximal independent set has the same size, or a graph in which every maximal independent set is maximum [5].

2. A Class of Planar Well-Covered Graphs with Girth Four:

A well covered graph is a graph in which every maximal independent set is a maximum independent set. The notion of a 1-well-covered graph was introduced by staples in her dissertation. A well- covered graph G is 1-well covered if and only if G-v is a is also well-covered for every point v in G. Expect for K_2 and C_5, every 1-well covered graph contains triangles or 4 – cylcles [6]. Thus, triangle – free 1-well covered graphs necessarily have girth 4. We show that all planar 1- well covered graphs of girth 4 belong to a specific infinite family, and we give a characterization of this family [11].

2.1 Theorem:

If $G \in W_2$ and v is a point on a 4-cycle in G, then $\deg(v) \geq 3$.

Proof:

Suppose, V is on the 4-cycle vabc in G. Also suppose that deg (v) = 2.
Then (v) and (b) cannot be extended to disjoint maximum independent sets in G,
A contradiction, Since $G \in W_2$. Thus, $\deg(v) \geq 3$.
In next theorem, We show that any point in a W_2 graph that is not on a triangle must be on a 5-cycle.

Hence Proof

2.2 Theorem:

Let $n \geq 3$ be a positive integer. Then there exists a planar W2 graph of girth 4, denoted G_n, such that $\alpha(G_n) = n$ and $|V(G_n)| = 3n-1$

Proof:

(By induction on n.) For n = 3, Let G_3 be the graph on eight points given in Figure 1. Then $\alpha(G_3) = 3$ and $|V(G_3)| = 3(3)- 1$. For $k \geq 3$, let G_{k+1}be a graph obtained from G_k by the construction given in Construction 1.

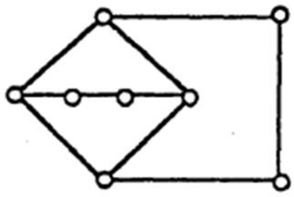

Figure 1

Assume $\alpha(G_k) = k$ and $|V(G_{k+1})| = 3k$

From the observation by known theorem, graph Gk+I is a planar W_2 graph of girth4.

Also, $|V(G_{k+1})| = |V(G_k)|+ 3 = 3k - 1 + 3 = 3(k+1) -1$, and $\alpha(G_{k+I}) = \alpha(C_k) + I =k +1$

Therefore, G_{k+1} satisfy the statement of the theorem. The result follows by induction.

Now that we have an infinite family of planar W_2 graphs of girth4, we work toward showing that all planar W_2 graphs of girth 4 are in the family in this theorem.

A characterization. Since the smallest cycle in a W_2 graph of girth 4 is a 4-cycle, it is of interest to learn we can about 4-cycles in these graphs.

Hence the proof

3. Well-Covered Graphs and Extend Ability:

We give general conditions for a graph to be well-covered which depend on the relationship between clique covers and independent sets [12]. A family of graphs closed under the operation of taking an induced sub graph, such that there exist polynomial-time algorithms for finding the independence number and the clique cover number. If a perfect graph with clique size bounded by an integer [7]. There is a polynomial-time algorithm for checking whether graph is well-covered. A graph is the join of a complete graph and a graph that is both 1-extendable and 2-extendable.

3.1 Theorem:

If a graph G is 2-extendable, then either G is both 1-extendable and 2-extendable or G is the join of a complete graph and a graph that is both 1-extendableand 2-extendable.

Proof:

Let G be 2-extendable and not 1-extendable. There exists a vertex v of G that is in no maximum independent set. Since G is 2-extendable, the vertex v must be adjacent to every other vertex in G. Let H be the subgraph of G induced by the set of vertices that are not in any maximum independent set.

The graph H is a complete graph and is joined to the remainder of G. We only need to show that G-H is a 1-extendable graph. From the definition of H it follows that each vertex of G-H is contained in a maximum independent set of G.

It also follows from the definition of H that every maximum independent set of G is contained in G-H, and hence is a maximum independent set of G-H.

So G-H is I-extendable.

We need the following two lemmas to prove Corollary 3.4. However, these lemmas are of independent interest because they establish a connection between l-extendability and Hall's condition and hence perfect matchings.

Hence proof

3.2 Corollary:

If a graph G has no isolated vertices and $\alpha(G) =1/2|V|$, then the following are equivalent:

1. G is well-covered.
2. G is both 1-extendable and 2-extendable.
3. G has a perfect matching and for every edge (u,v) of the perfect matching, G[E(N(u), N(v))] is a complete bipartite graph.
4. G has a perfect matching and for every edge (u, v) of the perfect matching and independent set S of G-u-v, at least one of the vertices u, v has no neighbor in S.

Proof:

Both (1) and (2) imply that G has a perfect matching by known Lemma. Now apply, by known theorem with the perfect matching as the clique cover where $t =\alpha$ and $d = 0$. It is immediate that (1) and (2) above are equivalent to (1) and (2) of Theorem

2.1. With slightly more work, we see that both (3) and (4) above are equivalent to (3) of known theorem.

Hence proof

4. Very Well Covered Graphs:

A simple graph G(V, E), is quasiregularizable if and only if, for every irredundant set I of G, there exists a matching of all of I into V- I. For every quasiregularizable graph G(V,E) of order n, IR≤½n. If IR =½n, then ⌈ = IR. For a simple graph G(V,E), the following properties are equivalent: (i) G is very well covered. (ii) There exists a perfect matching in G which satisfies the property (P). (iii) There exists at least one perfect matching in G, and every perfect matching of G satisfies (P). In every well covered graph C (V, E), we have $\Gamma\{C[\Gamma(A)]\} = \Gamma(A)$ for every perfect matching C and every $A \subset V$. In a very well covered graph, for every perfect matching and every point x. $\Gamma(x) \cap C[\Gamma(x)] = \emptyset$ and C [I' (x) is a stable set[8].

4.1 Theorem:

G_i is the same for every choice of a perfect matching in G.

Proof:

If C is a perfect matching in G and X a class, it is sufficient to show that every other perfect matching in G matches X with C(X) (not necessarily in the same way as C).
Let x be a vertex of G, X the class of x for a perfect matching C.
As $\Gamma\{C[\Gamma(X)]\} = \Gamma(X)$ (by known Corollary) and $|\Gamma'(X)| = |c[\Gamma(x)]$, every other perfect matching of G matches also the points of $\Gamma(X)$ with those of C[I' (X)], and in particular the points of C(X), included in $\Gamma(X)$, with points of C[I' (X)]
But, as no point of C(X) is joined to $C[\Gamma(X)]$-X, every perfect matching of G matches X with C(X).

Hence proof

Theorem 4.2:

The following properties are equivalent (i) G is a very well covered irreducible. (ii) G has a perfect matching C which satisfies the properties (P) and P_5: $x \in \Gamma(y) \Rightarrow C(x) \notin \Gamma[C(y)]$. (iii) G has a unique perfect matching and this matching satisfies (P).

Remark:

P_5 means that G does not contain a cycle C_4 with two edges of the matching C.

Proof:

(i) $\Rightarrow$ (ii). This is a consequence of the properties P_i.
(ii) $\Rightarrow$ (iii). If C satisfies P_5, each class consists of a unique vertex. From the by known proof of, every perfect matching matches X with C(X) and then G has a unique perfect matching.
(iiii)$\Rightarrow$(i).by known theorem shows that G is very well covered. If G has a unique perfect matching, each class consists of a unique point and then G is irreducible.

Hence proof

5. Strongly Well-Covered Graphs:

If G $\neq$ K2 is well-covered and e is a critical line in G, then G-e is not well covered. Suppose G is well-covered. Also suppose that S is an independent set and x is a point in G such that (i) $x \notin S$ and $x \sim v$ for exactly one v in S, and (ii) S dominates N[x]. Then G-e is not well-covered, where e = vx. Suppose G is strongly well-covered with $\alpha(G) > 2$. Then every point in G must have at least two nonadjacent neighbors [9]. Suppose e = uv is a line in a well-covered graph G such that G-e is not well covered. Then either (i) e is a critical line and there exists a maximum independent set I containing {u,v) in G-e, or (ii) e is not a critical line and there exists a maximal independent set J containing {u,v} in G-e such that |J| < α(G). If G is strongly well-covered and G is not complete, then $G_v$is strongly well-covered for all points v in G[10].

5.1 Lemma:

If G ≠ K2 is well-covered and e is a critical line in G, then G-e is not well covered.

Proof:

Let e = uv. Since G ≠ K2, then (without loss of generality) there exists some point a ~u, a ≠ v.

Since G is well-covered, there exists maximum independent set J in G such that a ∈ J.

In the graph G-e, the set J is maximal independent. Thus, G-e has a maximal independent set of size α (G). Since e is a critical line, α (G-e) = α(G) + 1.

Hence, the graph G-e is not well-covered. Note that as a consequence of Lemma 1, we have the statement that a strongly well covered graph (other than K2) has no critical lines. Thus, if G ≠K2 is strongly well covered, then α(G-e) = α(G) for all lines e in G.

If x is a point in a graph G, then the closed neighborhood of x is given by N[x] and consists of x and all its neighbors. The next two lemmas will be very helpful in eliminating candidate graphs as we develop the structure of strongly well-covered graphs.

Hence proof

5.2 Theorem:

Suppose G is well-covered. Also suppose that S is an independent set and x is a point in G such that (i) x ∉ S and x ~ v for exactly one v in S, and (ii) S dominates N[x]. Then G-e is not well-covered, where e = vx.

Proof:

Since G is well-covered and S is independent, then there exists maximum independent set J ⊇ S in G. Since v is in S and x ~ v, then x ∉ J.

Since S dominates N[x] and J ⊇S, Then N(x) ⋂ J = {v}. Thus, in the graph G-vx, the set ⋃ {x} is independent.

Hence, vx is a critical line in G. By known lemma, the graph G-vx is not well-covered.

Hence proof

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