International Journal of Multidisciplinary Research and Modern Education (IJMRME)
ISSN (Online): 2454 - 6119
(www.rdmodernresearch.com) Volume II, Issue II, 2016

BALANCED HESITANCY FUZZY GRAPHS

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Abstract:

In this paper, a new Hesitancy Fuzzy Graph Model called Balanced Hesitancy Fuzzy Graphs (BHFGs) is introduced. Also, we discuss the various concepts related with Balanced Hesitancy Fuzzy Graphs with their graphical representations. Further we develop the concept of self complementary hesitancy fuzzy graphs along with the theoretical illustration.

Key Words: Hesitancy Fuzzy Graphs, Hesitancy Subgraphs, Self Complementary Hesitancy Fuzzy Graph, Density of Hesitancy Fuzzy Graphs & Balanced Hesitancy Fuzzy Graphs.

1. Introduction:

Fuzzy Set Theory has its root in the eastern logic. It mainly deals with the ambiguity prevailing in the system. Lotfi A. Zadeh [24] in the year 1965 developed the theoretical framework of fuzzy set theory in order to capture the vagueness that has occurred in the real life circumstances. Then many ideas/concepts [7] have been developed and researchers throughout the world are coming out with numerous results by incorporating fuzzy set theoretical concepts in their experiments. Several hypotheses have been framed and suggestions are derived with the help of fuzzy decision making tools [18] to improve the decisions made by the decision makers, planners and other authorities. Also various concepts of fuzzy set theory have been successfully applied in other areas which include Medicine, Engineering, Economics, Robotics, Social studies and so on. Fuzzy Graph is one such concept that was first introduced by A. Rosenfeld [19] in the year 1975 and has been much useful in the field of operations research, system analysis, automata theory, signal processing and so on. Importantly, J. N. Mordeson [8], M. S. Sunitha [21] [22] and A. Nagoor Gani [15] defined other major concepts in the fuzzy graph theory. T. Pathinathan and J. Jesintha Rosline introduced two new fuzzy graphs namely Double Layered Fuzzy Graphs (DLFGs) [5] [9] [10] [16] [17] and extensively studied their important properties with application.

T. Pathinathan and J. Jon Arockiaraj [12] introduced a new fuzzy graph called Hesitancy Fuzzy Graphs and discussed their various theoretical properties and validations. In addition to this, the concept of regularity [15], constant [13], index matrix representation [14] and various Cartesian products [11] were also derived. This article presents the concepts of balanced extension of hesitancy fuzzy graphs and self complementary hesitancy fuzzy graphs with the help of theoretical illustrations.

Erdos and Renyi [3] first studied the balanced extension on random graphs [4] in order to deal with the complex networks. Complex networks in the sense with more connections and dimensions, the network become vague and it complicates the situation. To overcome such situation, Balanced Fuzzy Graphs (BFGs) is developed. T.AL-Hawary [1] introduced the concept of Balanced Fuzzy Graphs (BFGs) and further extensions have been made by Mohammed Akram and M. G. Karunambigai [6]. Karunambigai and others [6] defined the concepts of density and balanced notation for an Intuitionistic Fuzzy Graphs.

The article is organized as follows. Section 2 focuses on the basic concepts and notations of Fuzzy Graphs (FGs), Hesitancy Fuzzy Graphs (HFGs). Section 3 introduces the concept of Balanced Hesitancy Fuzzy Graphs (BHFGs) along with the illustration. Section 4 gives the theoretical validations and proofs for the newly introduced Self Complementary Hesitancy Fuzzy graphs (SCHFGs), which followed by conclusion in section 5.

2. Basic Definitions and Terminologies:

This section contains some basic definitions and examples on Hesitancy Fuzzy Graphs and its related topics.

Definition 2.1: Fuzzy Graph (FG) Let V be a non empty set. A fuzzy graph is a pair of functions $G(\mu_1,\mu_2)$ where μ_1 is a fuzzy subset of V, μ_2 is a symmetric fuzzy relation on μ_1.

$\mu_1 : V \to [0,1]$

$\mu_2 : V \times V \to [0,1]$

such that $\mu_2(v_i,v_j) \le \mu_1(v_i) \wedge \mu_1(v_j)\ \forall v_i,v_j \in V$.

The underlying crisp graph of the fuzzy graph $G(\mu_1,\mu_2)$ is denoted as $G^* : (\mu_1^*, \mu_2^*)$ where μ_1^* is referred to as the nonempty set V of nodes and $\mu_2^* = E \subseteq V \times V$. The crisp graph (V, E) is a special case of the fuzzy graph G with each vertex and edge of (V,E) having degree of membership 1.

Definition 2.2: Partial Fuzzy Subgraph: A fuzzy graph $H:(V',E')$ with $\mu_1':V'\to[0,1]$ and $\mu_2':V'\times V'\to[0,1]$ is said to an partial fuzzy subgraph of $G:(V,E)$ if,

(i) $V' \subseteq V$, where $\mu_1' \le \mu_1$, for all $v_i \in V',\ i=1,2,3,\dots,n.$

(ii) $E' \subseteq E$, where $\mu_2' \le \mu_2$, for all $(v_i,v_j) \in E',\ i,j=1,2,3,\dots,n.$

Definition 2.3: Fuzzy Subgraph: In Particular, a partial fuzzy subgraph $H:(V',E')$ is said to be a fuzzy subgraph of $G:(V,E)$ if,

(i) $V' \subseteq V$, where $\mu_1' = \mu_1$, for all $v_i \in V',\ i=1,2,3,\dots,n.$

(ii) $E' \subseteq E$, where $\mu_2' = \mu_2$, for all $(v_i,v_j) \in E',\ i,j=1,2,3,\dots,n.$

Definition 2.4: Density of a Fuzzy Graph [1]: The density of a fuzzy graph $G:(V,E)$ with $\mu_1:V\to[0,1]$ and $\mu_2:V\times V\to[0,1]$ is defined as,

$$D(G) = 2\left(\frac{\sum_{v_i,v_j\in V} \mu_2(v_i,v_j)}{\sum_{v_i,v_j\in V} \left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}\right).$$

Definition 2.5: Balanced Fuzzy Graph [1]: A fuzzy Graph $G(V,E)$ is said to be balanced fuzzy graph, if $D(H) \le D(G)$ for all fuzzy non-empty subgraphs $H:(V',E')$ of $G(V,E)$.

Definition 2.6: Strictly Balanced Fuzzy Graph [1]: A fuzzy graph $G=(V,E)$ is strictly balanced fuzzy graph if $D(H)=D(G)$ for all non-empty subgraphs $H:(V',E')$ of $G(V,E)$.

Definition 2.7: Hesitancy Fuzzy Graph (HFG) [12]: A Hesitancy Fuzzy Graph is of the form G = (V,E), where

(i) $V=\{v_1,v_2,...v_n\}$ such that $\mu_1:V\rightarrow[0,1], \gamma_1:V\rightarrow[0,1]$ and $\beta_1:V\rightarrow[0,1]$ denote the degree of membership, non-membership and hesitancy of the element $v_i\in V$ respectively and $\mu_1(v_i)+\gamma_1(v_i)+\beta_1(v_i)=1\ \forall v_i\in V$, Where $\beta_1(v_i)=1-[\mu_1(v_i)+\gamma_1(v_i)]$ and $0\le\mu_1(v_i)+\gamma_1(v_i)\le 1$ --------(1)

(ii) $E\subseteq V\times V$ where $\mu_2:V\times V\rightarrow[0,1], \gamma_2:V\times V\rightarrow[0,1]$ and $\beta_2:V\times V\rightarrow[0,1]$ are such that, $\mu_2(v_i,v_j)\le\min(\mu_1(v_i),\mu_1(v_j))$ -------(2)

$\gamma_2(v_i,v_j)\le\max(\gamma_1(v_i),\gamma_1(v_j))$ -------(3)

$\beta_2(v_i,v_j)\le\min(\beta_1(v_i),\beta_1(v_j))$ -------(4)

And $0\le\mu_2(v_i,v_j)+\gamma_2(v_i,v_j)+\beta_2(v_i,v_j)\le 1\forall(v_i,v_j)\in E$ --------(5)

Notations:

- ✓ $\langle v_i,\mu_1,\gamma_1,\beta_1\rangle$ denotes the vertex, degree of membership, non-membership and hesitancy of the vertex v_i.
- ✓ $\langle e_{ij},\mu_2,\gamma_2,\beta_2\rangle$ denotes the edge, degree of membership, non-membership and hesitancy of the edge relation $e_{ij}=(v_i,v_j)$ on V.

2.8 Definition: Complete Hesitancy Fuzzy Graph [12]

A HFG, G = (V,E) is said to be a *complete HFG* if,

$$\mu_2(v_i,v_j)=\min(\mu_1(v_i),\mu_1(v_j))$$

$$\gamma_2(v_i,v_j)=\max(\gamma_1(v_i),\gamma_1(v_j))$$

$$\beta_2(v_i,v_j)=\min(\beta_1(v_i),\beta_1(v_j))$$

2.9 Definition: Complement of Hesitancy Fuzzy Graph [12]

Let G = (V,E) be an HFG, then the *complement of the HFG* is a HFG, $\bar{G}(\bar{V},\bar{E})$ where

$$\bar{V}=V, (i.e.,)\ \bar{\mu}_1=\mu_{1i}; \bar{\gamma}_1=\gamma_1; \bar{\beta}_1=\beta_1$$

and

$$\bar{\mu}_2=\min(\mu_1,\mu_1)-\mu_2,$$

$$\bar{\gamma}_2=\min(\gamma_1,\gamma_1)-\gamma_2\ and$$

$$\bar{\beta}_2=\min(\beta_1,\beta_1)-\beta_2\ \forall v_i,v_j\in V$$

3. Balanced Hesitancy Fuzzy Graphs

This section presents the concept of subgraph, density and balanced notation in terms of Hesitancy Fuzzy Graphs [12]. Also this section provides the geometrical representation of the above mentioned concepts with theoretical illustration.

3.1 Definition: Partial Hesitancy Fuzzy Subgraph

An FG $H=(V',E')$ is said to be a partial hesitancy fuzzy subgraph of $G=(V,E)$ if

(i) $V'\subseteq V$, where $\mu_1'\le\mu_1$, $\gamma_1'\le\gamma_1$, $\beta_1'\le\beta_1$ for all $v_i\in V'$, $i=1,2,3,\ldots,n$.

(ii) $E'\subseteq E$, where $\mu_2'\le\mu_2$, $\gamma_2'\le\gamma_2$ $\beta_2'\le\beta_2$ for all $(v_i,v_j)\in E'$, $i,j=1,2,3,\ldots,n$.

3.2 Definition: Hesitancy Fuzzy Subgraph

An FG $H=(V',E')$ is said to be a hesitancy fuzzy subgraph of $G=(V,E)$ if

(i) $V'\subseteq V$, where $\mu_1'=\mu_1$, $\gamma_1'=\gamma_1$, $\beta_1'=\beta_1$ for all $v_i\in V'$, $i=1,2,3,\ldots,n$.

(ii) $E'\subseteq E$, where $\mu_2'=\mu_2$, $\gamma_2'=\gamma_2$, $\beta_2'=\beta_2$ for all $(v_i,v_j)\in E'$, $i,j=1,2,3,\ldots,n$.

3.3 Definition: Density of Hesitancy Fuzzy Graph

The density of an intuitionstic fuzzy graph $G=(V,E)$ is $D(G)=\left(D_\mu(G),D_\gamma(G),D_\beta(G)\right)$, where,

$D_\mu(G)$ is defined by $D_\mu(G)=2\left(\dfrac{\sum\limits_{v_i,v_j\in V}\mu_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}\right)$, for $v_i,v_j\in V$

$D_\gamma(G)$ is defined by $D_\gamma(G)=2\left(\dfrac{\sum\limits_{v_i,v_j\in V}\gamma_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\gamma_1(v_i)\vee\gamma_1(v_j)\right)}\right)$, for $v_i,v_j\in V$

$D_\beta(G)$ is defined by $D_\beta(G)=2\left(\dfrac{\sum\limits_{u,v\in V}\beta_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\beta_1(v_i)\wedge\beta_1(v_j)\right)}\right)$, for $v_i,v_j\in V$

Otherwise,

$$D(G)=\left(D_\mu(G),D_\gamma(G),D_\beta(G)\right)$$

$$D(G)=\left(2\left(\frac{\sum\limits_{v_i,v_j\in V}\mu_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}\right),2\left(\frac{\sum\limits_{v_i,v_j\in V}\gamma_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\gamma_1(v_i)\vee\gamma_1(v_j)\right)}\right),2\left(\frac{\sum\limits_{u,v\in V}\beta_2(v_i,v_j)}{\sum\limits_{(v_i,v_j)\in E}\left(\beta_1(v_i)\wedge\beta_1(v_j)\right)}\right)\right)$$

3.4: Definition: Balanced Hesitancy Fuzzy Graph (BHFG)

A hesitancy fuzzy graph $G=(V,E)$ is balanced if $D(H)\le D(G)$, that is, $D_\mu(H)\le D_\mu(G)$, $D_\gamma(H)\le D_\gamma(G)$ and $D_\beta(H)\le D_\beta(G)$ for all subgraphs *H* of *G*.

3.5 Illustration: Balanced Hesitancy Fuzzy Graph

Let $G(V,E)$ is a hesitancy fuzzy graph with the vertex set $V=\{V_1,V_2,V_3,V_4\}$ given by $V=(\mu_{1i},\gamma_{1i},\beta_{1i})$ where $\mu_{1i}:V\to[0,1];\gamma_{1i}:V\to[0,1];\beta_{1i}:V\to[0,1]$ denotes the degree of membership, non-membership and hesitancy degree. Then edge relation between them is given by

$$V\times V=\begin{Bmatrix}\{V_1,V_2\},\{V_1,V_3\},\{V_1,V_4\},\{V_2,V_3\},\{V_2,V_4\},\{V_3,V_4\},\\ \{V_1,V_2,V_3\},\{V_1,V_2,V_4\},\{V_1,V_3,V_4\},\{V_2,V_3,V_4\},\\ \{V_1,V_2,V_3,V_4\}\end{Bmatrix}$$

with $V\times V=(\mu_{2ij},\gamma_{2ij},\beta_{2ij})$ where $\mu_{2ij}:V\times V\to[0,1]$; $\gamma_{2ij}:V\times V\to[0,1]$; $\beta_{2ij}:V\times V\to[0,1]$ denotes the edge relation of membership, non-membership and hesitancy degree. The hesitancy fuzzy graph is figured as follows:

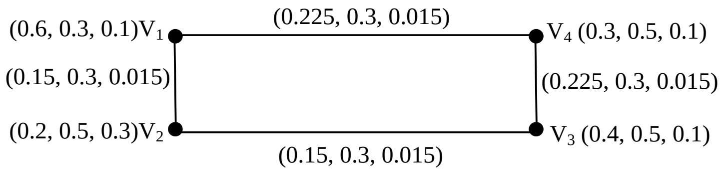


Figure 1: Balanced Hesitancy Fuzzy Graph

Calculation of density values for core Hesitancy Fuzzy Graph G (V,E):

Membership Density $D_\mu(G)=2\left(\frac{0.225+0.225+0.15+0.15}{0.3+0.3+0.2+0.2}\right)=1.5$

Non-membership Density $D_\gamma(G)=2\left(\frac{0.3+0.3+0.3+0.3}{0.5+0.5+0.5+0.5}\right)=1.2$

Hesitancy Density $D_\beta(G)=2\left(\frac{0.015+0.015+0.015+0.015}{0.1+0.1+0.1+0.1}\right)=0.3$

Calculation of density values for Hesitancy Fuzzy Subgraph (H_1): {V_1, V_4}

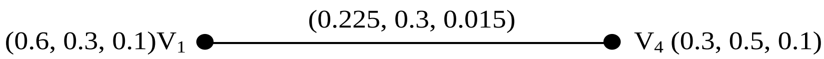


Figure 2: Hesitancy Fuzzy Subgraph H_1

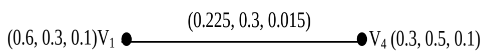
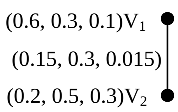
Membership Density $D_\mu(H_1)=2\left(\frac{0.225}{0.3}\right)=1.5$

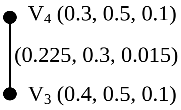
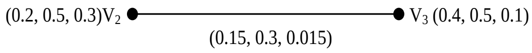
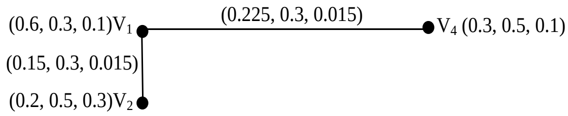
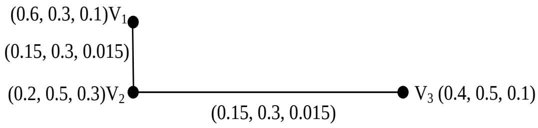
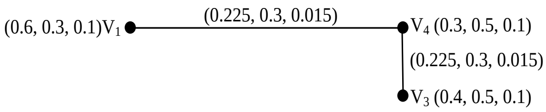
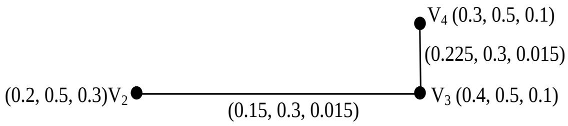
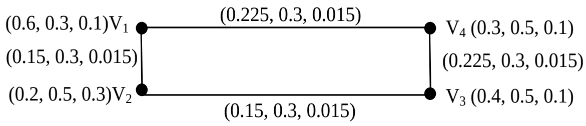
Non-membership Density $D_\gamma(H_1)=2\left(\frac{0.3}{0.5}\right)=1.2$

Hesitancy Density $D_\beta(H_1)=2\left(\frac{0.015}{0.1}\right)=0.3$

Similarly, the density values of the other subgraphs are found by using the definition (Definition 3.3) and it is shown in the below table (Table 1). Also from the figure (Figure 1) it is noted that the subgraphs {V_1, V_3} and {V_2, V_4} does not have an edge. Therefore the density value is (0,0,0).

Table 1: Density values of Balanced Hesitancy Fuzzy Graph

Notation	**Subgraph**	**Vertex Set**	$(D_\mu(G),D_\gamma(G),D_\beta(G))$
H_1		{V_1,V_4}	(1.5,1.2,0.3)
H_2		{V_1,V_3}	(0,0,0)
H_3		{V_1,V_2}	(1.5,1.2,0.3)

H_4		{V2,V3}	(1.5,1.2,0.3)
H_5		{V2,V4}	(0,0,0)
H_6		{V2,V3}	(1.5,1.2,0.3)
H_7		{V1,V2,V4}	(1.5,1.2,0.3)
H_8		{V1,V2,V3}	(1.5,1.2,0.3)
H_9		{V1,V3,V4}	(1.5,1.2,0.3)
H_{10}		{V2,V3,V4}	(1.5,1.2,0.3)
H_{11}		{V1,V2,V3,V4}	(1.5,1.2,0.3)

From the above table (Table 1), the density values of all the subgraphs are less than or equal to the density values of the core graph. Therefore the graph (Figure 1) is said to be a Balanced Hesitancy Fuzzy Graph.

3.6 Definition: Strictly Balanced Hesitancy Fuzzy Graph

A hesitancy fuzzy graph $G=(V,E)$ is said to be strictly balanced if $D(H)<D(G)$, that is, $D_\mu(H)<D_\mu(G)$, $D_\gamma(H)<D_\gamma(G)$ and $D_\beta(H)<D_\beta(G)$ for all subgraphs *H* of *G*.

3.7 Illustration: Strictly Balanced Hesitancy Fuzzy Graph

Let $G(V,E)$ is a hesitancy fuzzy graph with the vertex set $V=\{V_1,V_2,V_3\}$ given by $V=(\mu_{1i},\gamma_{1i},\beta_{1i})$ and the edge relation between them is given by with $V\times V=(\mu_{2ij},\gamma_{2ij},\beta_{2ij})$. For the following hesitancy fuzzy graph (Figure 3), the density values are tabulated (Table 2) as follows:

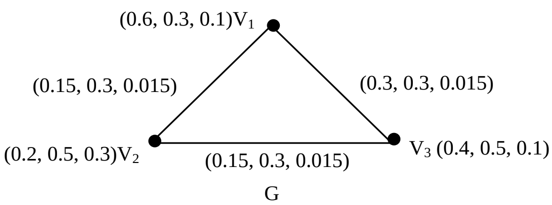


Figure 3: Strictly Balanced Hesitancy Fuzzy graph

Table 2: Density values of Strictly Balanced Hesitancy Fuzzy Graphs

Notation	Subgraph	$(D_\mu(G), D_\gamma(G), D_\beta(G))$
H_1	$\{V_1,V_2\}$	(1.5,1.2,0.3)
H_2	$\{V_1,V_3\}$	(1.5,1.2,0.3)
H_3	$\{V_2,V_3\}$	(1.5,1.2,0.3)
H_4	$\{V_1,V_2,V_3\}$	(1.5,1.2,0.3)

From the above table (Table 2), the density values of all the subgraphs are equal to the density values of the core graph (Figure 3). Therefore the graph (Figure 3) is said to be a Strictly Balanced Hesitancy Fuzzy Graph.

Based on the concept of completeness, the balanced notion of the hesitancy fuzzy graphs is proved by the following theorem. Also illustration followed by the theorem gives the validation for its proof.

3.8 Theorem:

Every complete hesitancy fuzzy graph is balanced.

Proof:

Let $G=(V,E)$ be a complete HFG, then by the definition of complete HFG, we have

$\mu_2(v_i,v_j)=\min(\mu_1(v_i),\mu_1(v_j))$; $\gamma_2(v_i,v_j)=\max(\gamma_1(v_i),\gamma_1(v_j))$; $\beta_2(v_i,v_j)=\min(\beta_1(v_i),\beta_1(v_j))$

for every $v_i,v_j\in V$.

Now, the density of a hesitancy fuzzy graph (Definition 3.3) is defined as follows;

$$D(G)=\left(\left(\frac{2\sum_{v_i,v_j\in V}\mu_2(v_i,v_j)}{\sum_{(v_i,v_j)\in E}\left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}\right),\left(\frac{2\sum_{v_i,v_j\in V}\gamma_2(v_i,v_j)}{\sum_{(v_i,v_j)\in E}\left(\gamma_1(v_i)\vee\gamma_1(v_j)\right)}\right),\left(\frac{2\sum_{v_i,v_j\in V}\beta_2(v_i,v_j)}{\sum_{(v_i,v_j)\in E}\left(\beta_1(v_i)\wedge\beta_1(v_j)\right)}\right)\right)$$

From the definition of complete hesitancy fuzzy graph (Definition 2.8) the above equation is written as follows;

$$D(G)=\left(\left(\frac{2\sum_{v_i,v_j\in V}\left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}{\sum_{(v_i,v_j)\in E}\left(\mu_1(v_i)\wedge\mu_1(v_j)\right)}\right),\left(\frac{2\sum_{v_i,v_j\in V}\left(\gamma_1(v_i)\vee\gamma_1(v_j)\right)}{\sum_{(v_i,v_j)\in E}\left(\gamma_1(v_i)\vee\gamma_1(v_j)\right)}\right),\left(\frac{2\sum_{v_i,v_j\in V}\left(\beta_1(v_i)\wedge\beta_1(v_j)\right)}{\sum_{(v_i,v_j)\in E}\left(\beta_1(v_i)\wedge\beta_1(v_j)\right)}\right)\right)$$

$$D(G)=(2,2,2)$$

Let H be a non empty subgraphs of G. In similar way, $D(H)=(2,2,2)\,\forall H\subseteq G$.

Thus, $D(G)=D(H)=(2,2,2)$

Thus G is balanced.

3.8 Illustration: Every complete hesitancy fuzzy graph is balanced

Let $G(V,E)$ is a complete hesitancy fuzzy graph with the vertex set $V=\{V_1,V_2,V_3,V_4\}$ given by $V=(\mu_{1i},\gamma_{1i},\beta_{1i})$ where $\mu_{1i}:V\to[0,1];\gamma_{1i}:V\to[0,1];\beta_{1i}:V\to[0,1]$ denotes the degree of membership, non-membership and hesitancy degree. Then edge relation between them is given by with $V\times V=(\mu_{2ij},\gamma_{2ij},\beta_{2ij})$ where $\mu_{2ij}:V\times V\to[0,1]$; $\gamma_{2ij}:V\times V\to[0,1]$; $\beta_{2ij}:V\times V\to[0,1]$ denotes the edge relation of membership, non-membership and hesitancy degree.

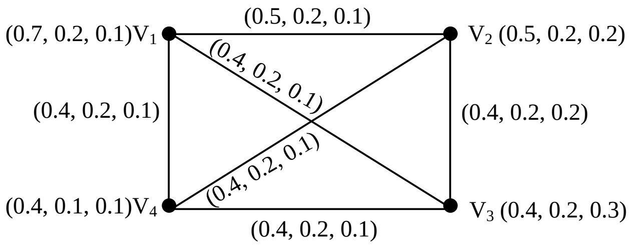


Figure 4: Complete Hesitancy Balanced Fuzzy Graph

Calculation of density values for core Hesitancy Fuzzy Graph G (V,E):

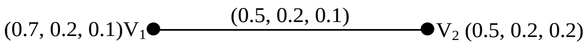
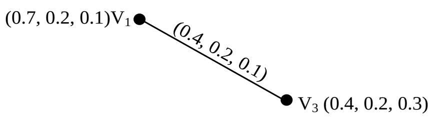
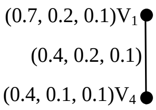
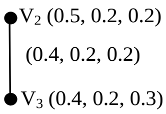
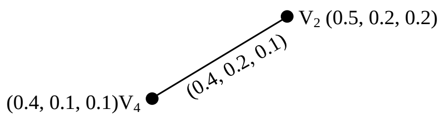
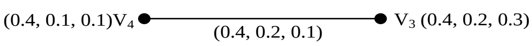
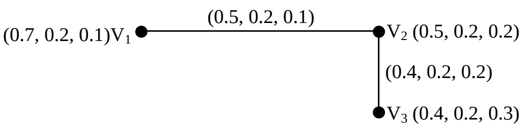
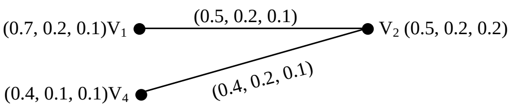
By using the definition (Definition 3.3), the density values of the above graph are calculated. The density values of the graph G(V,E) is given by;

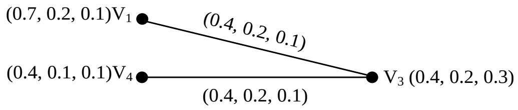
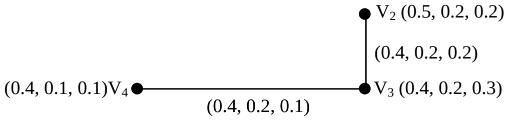
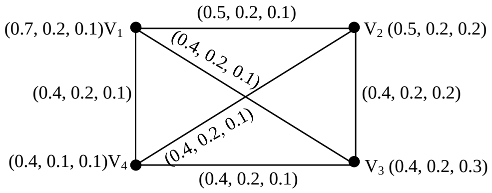
$$D(G) = \left(D_\mu(G), D_\gamma(G), D_\beta(G)\right)$$

$$D(G) = (2,2,2)$$

The density values for the subgraphs are given in the below table (Table 3). And it is noted that, the density values of all the subgraphs has the density value as (2,2,2). Therefore it proves the above theorem and also the graph is said to be a complete balanced hesitancy fuzzy graph.

Table 3: Density Values of Complete Hesitancy Balanced Fuzzy Graphs

Notation	**Subgraph**	**Vertex Set**	$\left(\mathbf{D}_\mu(\mathbf{G}), \mathbf{D}_\gamma(\mathbf{G}), \mathbf{D}_\beta(\mathbf{G})\right)$
H_1		$\{V_1,V_2\}$	(2,2,2)
H_2		$\{V_1,V_3\}$	(2,2,2)
H_3		$\{V_1,V_4\}$	(2,2,2)
H_4		$\{V_2,V_3\}$	(2,2,2)
H_5		$\{V_2,V_4\}$	(2,2,2)
H_6		$\{V_3,V_4\}$	(2,2,2)
H_7		$\{V_1,V_2,V_3\}$	(2,2,2)
H_8		$\{V_1,V_2,V_4\}$	(2,2,2)

H_9		$\{V_1,V_3,V_4\}$	(2,2,2)
H_{10}		$\{V_2,V_3,V_4\}$	(2,2,2)
H_{11}		$\{V_1,V_2,V_3,V_4\}$	(2,2,2)

4. Self-Complimentary Hesitancy Fuzzy Graph:

4.1 Definition: Self-Complimentary Hesitancy Fuzzy Graph:

A hesitancy fuzzy graph G is self complementary if $G=\bar{G}$.

4.2 Proposition:

Let G (V,E) with $\mu : V \rightarrow [0,1]$ and $\mu_2 : V\times V \rightarrow [0,1]$ be a self-complimentary fuzzy graph,. Then we have,

$$\sum_{v_i,v_j\in V}\left(\mu_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{v_i,v_j\in V}\min\left(\mu_1\left(v_i\right),\mu_1\left(v_j\right)\right)$$

$$\sum_{v_i,v_j\in V}\left(\gamma_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{v_i,v_j\in V}\max\left(\gamma_1\left(v_i\right),\gamma_1\left(v_j\right)\right)$$

$$\sum_{v_i,v_j\in V}\left(\mu_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{v_i,v_j\in V}\min\left(\beta_1\left(v_i\right),\beta_1\left(v_j\right)\right)$$

Proof:

Let G (V,E) be a self complimentary fuzzy graph, then we have

$\bar{V}=V_i$ i.e., $\bar{\mu}_1=\mu_1$; $\bar{\gamma}_1=\gamma_1$; $\bar{\beta}_1=\beta_1$ and

$$\bar{\mu}_2=\min\left(\mu_1\left(v_i\right),\mu_1\left(v_j\right)\right)-\mu_2$$

$$\bar{\mu}_2+\mu_2=\min\left(\mu_1\left(v_i\right),\mu_1\left(v_j\right)\right)$$

$$2\mu_2=\min\left(\mu_1\left(v_i\right),\mu_1\left(v_j\right)\right)$$

$$\mu_2=\frac{1}{2}\min\left(\mu_1\left(v_i\right),\mu_1\left(v_j\right)\right)$$

Similarly;

$$\gamma_2=\frac{1}{2}\max\left(\gamma_1\left(v_i\right),\gamma_1\left(v_j\right)\right)$$

$$\beta_2=\frac{1}{2}\min\left(\beta_1\left(v_i\right),\beta_1\left(v_j\right)\right)$$

4.3 Illustration: Self complimentary Hesitancy Fuzzy Graph:

Let G (V,E) be a hesitancy fuzzy graph with $V=\{V_1,V_2,V_3\}$ given by $V=\left(\mu_{1i},\gamma_{1i},\beta_{1i}\right)$ and the edge relation between them is given by with $V\times V=\left(\mu_{2ij},\gamma_{2ij},\beta_{2ij}\right)$. Then $\bar{G}(\bar{V},\bar{E})$ is defined by using the definition (Definition 2.9):

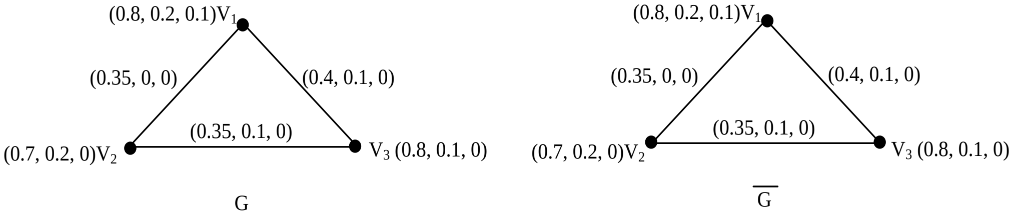


Figure 5: Self Complementary Hesitancy Fuzzy Graph

By using the above proposition, we have

$$\sum_{u,v\in V}\left(\mu\ (u,v)\right)=\frac{1}{2}\sum_{u,v\in E}\left(\mu(u)\wedge\mu\ (v)\right)$$

$$0.4+0.35+0.35 = \frac{1}{2}(0.8+0.7+0.7)$$

$$1 = 1$$

$$\sum_{u,v\in V}\left(\gamma\ (u,v)\right)=\frac{1}{2}\sum_{u,v\in E}\left(\gamma\ (u)\wedge\gamma\ (v)\right)$$

$$0.1+0.1+0.1 = \frac{1}{2}(0.2+0.2+0.2)$$

$$0.3 = 0.3$$

$$\sum_{u,v\in V}\left(\beta\ (u,v)\right)=\frac{1}{2}\sum_{u,v\in E}\left(\beta(u)\wedge\beta(v)\right)$$

$$0+0+0 = \frac{1}{2}(0+0+0)$$

That is $G=\overline{G}$.

Therefore G is self complementary.

4.4 Theorem:

Every self-complimentary hesitancy fuzzy graph has density equal to 1.

Proof:

Let G(V,E) be the self-complimentary hesitancy fuzzy graph. Then we have

$$\sum_{v_i,v_j\in V}\left(\mu_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{\left(v_i,v_j\right)\in E}\left(\mu_1\left(v_i\right)\wedge\mu_1\left(v_j\right)\right)$$

$$\sum_{v_i,v_j\in V}\left(\gamma_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{\left(v_i,v_j\right)\in E}\left(\gamma_1\left(v_i\right)\vee\gamma_1\left(v_j\right)\right)$$

$$\sum_{v_i,v_j\in V}\left(\beta_2\left(v_i,v_j\right)\right)=\frac{1}{2}\sum_{\left(v_i,v_j\right)\in E}\left(\beta_1\left(v_i\right)\wedge\beta_1\left(v_j\right)\right)$$

Also we have

$$D_\mu(G)=2\left(\frac{\sum_{v_i,v_j\in V}\mu_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\mu_1\left(v_i\right)\wedge\mu_1\left(v_j\right)\right)}\right)$$

$$D_\gamma(G)=2\left(\frac{\sum_{v_i,v_j\in V}\gamma_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\gamma_1\left(v_i\right)\wedge\gamma_1\left(v_j\right)\right)}\right)$$

$$D_\beta(G)=2\left(\frac{\sum_{v_i,v_j\in V}\beta_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\beta_1\left(v_i\right)\wedge\beta_1\left(v_j\right)\right)}\right).$$

The density of a hesitancy fuzzy graph is given by,

$$\left(D_\mu\left(G\right),D_\gamma\left(G\right)\right),D_\beta\left(G\right)=2\left(\left(\frac{\sum_{v_i,v_j\in V}\mu_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\mu_1\left(v_i\right)\wedge\mu_1\left(v_j\right)\right)}\right),\left(\frac{\sum_{v_i,v_j\in V}\gamma_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\gamma_1\left(v_i\right)\wedge\gamma_1\left(v_j\right)\right)}\right),\left(\frac{\sum_{v_i,v_j\in V}\beta_2\left(v_i,v_j\right)}{\sum_{v_i,v_j\in V}\left(\beta_1\left(v_i\right)\wedge\beta_1\left(v_j\right)\right)}\right)\right)$$

$$=2\left(\frac{\sum_{v_i,v_j\in V}\mu_2\left(v_i,v_j\right)}{2\sum_{v_i,v_j\in V}\mu_2\left(v_i,v_j\right)},\frac{\sum_{v_i,v_j\in V}\gamma_2\left(v_i,v_j\right)}{2\sum_{v_i,v_j\in V}\gamma_2\left(v_i,v_j\right)},\frac{\sum_{v_i,v_j\in V}\beta_2\left(v_i,v_j\right)}{2\sum_{v_i,v_j\in V}\beta_2\left(v_i,v_j\right)}\right)$$

$$=2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$

$$=\left(1,1,1\right)$$

Hence proved

4.5 Illustration: Every Self-Complimentary Hesitancy Fuzzy Graph Has Density Equal To 1

For the graph defined below, the density values are calculated using the definition (Definition 3.3). The values in the below table (Table 4) shows the graph is self complementary hesitancy fuzzy graph with the density values be (1,1,1).

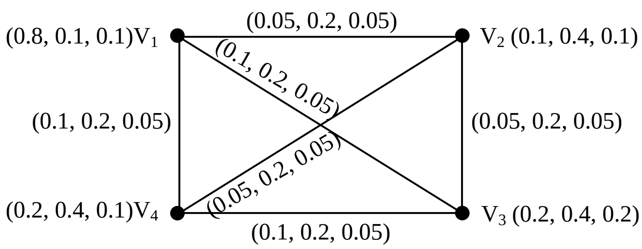
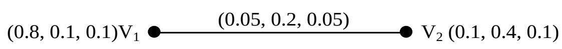
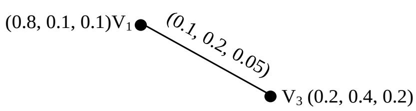
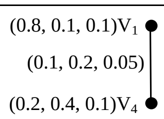
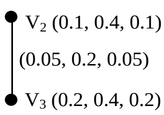
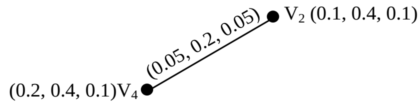
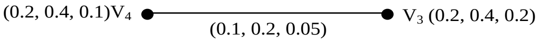
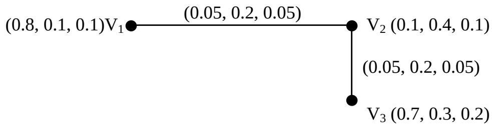
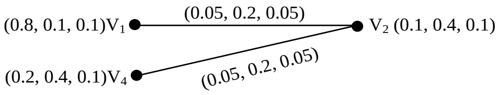
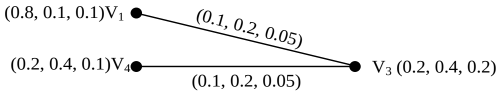
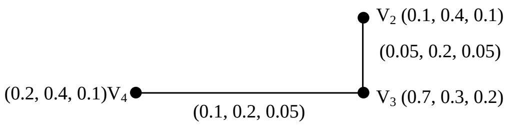
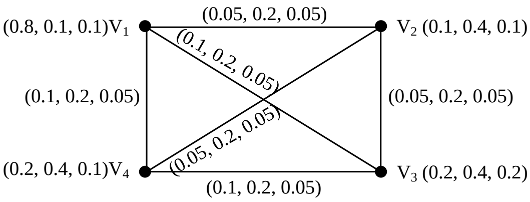


Figure 6: Self-Complementary Hesitancy Fuzzy Graph with Density (1,1,1)

Table 4: Self Complementary Balanced Hesitancy Fuzzy Graph with Density Value 1

Notation	Subgraph	Vertex Set	$\left(D_\mu\left(G\right),D_\gamma\left(G\right),D_\beta\left(G\right)\right)$
H_1		{V_1,V_2}	(1,1,1)
H_2		{V_1,V_3}	(1,1,1)
H_3		{V_1,V_4}	(1,1,1)
H_4		{V_2,V_3}	(1,1,1)

H_5		$\{V_2,V_4\}$	(1,1,1)
H_6		$\{V_3,V_4\}$	(1,1,1)
H_7		$\{V_1,V_2,V_3\}$	(1,1,1)
H_8		$\{V_1,V_2,V_4\}$	(1,1,1)
H_9		$\{V_1,V_3,V_4\}$	(1,1,1)
H_{10}		$\{V_2,V_3,V_4\}$	(1,1,1)
H_{11}		$\{V_1,V_2,V_3,V_4\}$	(1,1,1)

5. Conclusion:

Through this paper, the Balanced Hesitancy Fuzzy graphs (BHFGs) is introduced and studied briefly with the significant theoretical results. The concept of Hesitancy Fuzzy Subgraph, Partial Hesitancy Fuzzy Subgraph and balanced extension of hesitancy fuzzy graphs are developed. In addition to that, Self Complementary Hesitancy Fuzzy Graphs (SCHFGs) is discussed and the special case where it has density (1, 1, 1) is theoretically proved and verified with the proper illustration.

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