\( \tau_1 \tau_2 \)-FB-OPEN SETS IN FINE - BITOPOLITICAL SPACES

S. Kalaiselvi* & M. P. Sindhu**

*Associate Professor, Department of Mathematics, Sri G.V.G Visalakshi College for Women (Autonomous), Udumalpet, Tamilnadu

**M.Phil Scholar, Department of Mathematics, Sri G.V.G Visalakshi College for Women (Autonomous), Udumalpet, Tamilnadu

Abstract:
In this paper, fine-bitopological spaces and a new class of open sets in it are defined. Also some of their basic properties are discussed.

Index Terms: \( \tau_i \) fine-open sets (for \( i = 1, 2 \)), \( \tau_i \tau_2 \) fine-open sets, \( \tau_{i,2} \) fine-open sets, \( \tau_i \tau_2 \) fb-open sets & \( \tau_2 \tau_1 \) fb-open sets.

1. Introduction:
Topography plays a key role in many fields of information systems, particle physics, quantum physics and high energy physics and several branches of mathematics such as data mixing, computational topology for geometric design, computer aided geometric design and digital topology. In 1963, Kelly [4] introduced the notion of bitopological spaces; the ordered triple \( (X, \tau_1, \tau_2) \) where \( \tau_1 \) and \( \tau_2 \) are topologies on \( X \) is called bitopological space. Andrijev [1] introduced the concept of \( b \)-open sets in topological spaces in 1996. Later Ab Khadra and Nasef discussed \( b \)-open sets in bitopological spaces in 2003. Many researchers studied characterization of bitopological spaces [2, 3]. In 2012, Powar and Rajak [5] have investigated a special case of generalized topological space called fine topological space.

2. Preliminaries:
Throughout the paper \( (X, \tau) \) and \( (X, \tau_1, \tau_2) \) denote topological space and bitopological space without separation axioms unless specified. Let \( A \subseteq X \). The closure and the interior of a subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) with respective to \( \tau_i \) (\( i = 1, 2 \)) will be denoted by \( \tau_i \cdot \text{cl}(A) \) and \( \tau_i \cdot \text{int}(A) \). By \( (i, j) \) we mean the pair of topologies (\( \tau_i, \tau_j \)) for \( i, j = 1, 2 \) and \( i \neq j \).

Definition 2.1 [3]: A subset \( A \) of a bitopological space \( (X, \tau_1, \tau_2) \) is said to be
(i) \( (i, j) \)-b-open if \( A \subseteq \tau_i \cdot \text{int}(\tau_j \cdot \text{cl}(A)) \cup \tau_j \cdot \text{cl}(\tau_i \cdot \text{int}(A)) \).
(ii) \( (i, j) \)-b-closed if \( A \subseteq \tau_i \cdot \text{int}(\tau_j \cdot \text{cl}(A)) \cap \tau_j \cdot \text{cl}(\tau_i \cdot \text{int}(A)) \).

Definition 2.2 [3]: Let \( A \) be a subset of a bitopological space \( (X, \tau_1, \tau_2) \).
(i) The \( (i, j) \)-b-closure of \( A \) denoted by \( (i, j) \)-b-cl\( (A) \), is defined by the intersection of all \( (i, j) \)-b-closed sets containing \( A \).
(ii) The \( (i, j) \)-b-interior of \( A \) denoted by \( (i, j) \)-b-int\( (A) \), is defined by the union of all \( (i, j) \)-b-open sets contained in \( A \).

Definition 2.3 [2]: \( A \) is called \( \tau_1 \tau_2 \)-open if \( A \in \tau_1 \cup \tau_2 \), the complement of \( \tau_1 \tau_2 \)-open set is called \( \tau_1 \tau_2 \)-closed set.

Definition 2.4 [2]: \( A \) is called \( \tau_{1,2} \)-open if \( A = A_1 \cup B_1 \), where \( A_i \in \tau_1 \), \( B_i \in \tau_2 \), the complement of \( \tau_{1,2} \)-open set is called \( \tau_{1,2} \)-closed set.

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Remark 2.5 [2]: Note that \( \tau, \tau \)-open set of \((X, \tau, \tau)\) need not necessarily form a topology on \((X, \tau, \tau)\).

**Definition 2.6 [5]:** Let \((X, \tau)\) be a topological space we define 
\[ \tau_a = \{ G_a(\neq X) : G_a \subset X, G_a \cap A_a \neq \phi, \text{ for } A_a \in \tau \text{ and } A_a \neq \phi, X, \text{ for some } \alpha \in J, \text{ where } J \text{ is the index set}. \]

Now, we define \( \tau_j = \{ \phi, X, \bigcup_{\alpha \in j} \{ \tau_a \} \} \)
The above collection \( \tau_j \) of subsets of \( X \) is called the fine collection of subsets of \( X \) and \((X, \tau, \tau_j)\) is said to be the fine space \( X \) generated by the topology \( \tau \) on \( X \).

**Definition 2.7 [5]:** A subset \( U \) of a fine space \( X \) is said to be fine-open sets of \( X \), if \( U \) belongs to the collection \( \tau_f \) and the complement of every fine-open sets of \( X \) is called the fine-closed sets of \( X \) and we denote the collection by \( F_f \).

**Definition 2.8 [5]:** Let \( A \) be the subset of a fine space \( X \), the fine interior of \( A \) is defined as the union of all fine-open sets contained in the set \( A \) i.e. the largest fine-open set contained in \( A \) and is denoted by \( f_{int} \).

**Definition 2.9 [5]:** Let \( A \) be the subset of a fine space \( X \), the fine closure of \( A \) is defined as the intersection of all fine-closed sets containing the set \( A \) i.e. the smallest fine-closed set containing the set \( A \) and is denoted by \( f_{cl} \).

**Remark 2.10 [5]:** Let \((X, \tau, \tau_j)\) be a fine space and \( A \) be any arbitrary subset of \( X \). Then:

1. \( \text{int}(A) \subseteq f_{int}(A) \)
2. \( f_{cl}(A) \subseteq \text{cl}(A) \)

3. **Fine-Bitopological Spaces:**

   In this section \( X \) denotes the fine-bitopological space \((X, \tau_1, \tau_2, \tau_f, \tau_f)\).

**Definition 3.1:** Let \((X, \tau_1, \tau_2)\) be a bi-topological space we define 
\[ \tau (A_{\alpha}) = \tau_{\alpha}(\text{say}) = \{ G_{\alpha}(\neq X) : G_{\alpha} \subset X, G_{\alpha} \cap A_{\alpha} \neq \phi, \text{ for } A_{\alpha} \in \tau_i \text{ and } A_{\alpha} \neq \phi, X, \text{ for some } \alpha \in J, \text{ where } J \text{ is the index set and for } i = 1, 2 \} . \]

Now, we define \( \tau_{i} = \{ \phi, X, \bigcup_{\alpha \in J} \{ \tau_{i \alpha} \} \} \), for \( i = 1, 2 \).

The above collection \( \tau_{i} = \{ \phi, X, \bigcup_{\alpha \in J} \{ \tau_{i \alpha} \} \} \), for \( i = 1, 2 \).

The complement of every \( \tau_i \)-open set of \( X \) is called the \( \tau_i \)-fine-open set of \( X \) for \( i = 1, 2 \), if \( U \) belongs to the collection \( \tau_f \).

**Definition 3.2:** A subset \( U \) of a fine-bi space \( X \) is said to be \( \tau_i \)-fine-open set of \( X \) for \( i = 1, 2 \), if \( U \) belongs to the collection \( \tau_f \).

**Definition 3.3:** The complement of every \( \tau_i \)-open set of \( X \) is called the \( \tau_i \)-fine-closed set of \( X \) for \( i = 1, 2 \) and this collection is denoted by \( \tau_f \).

**Definition 3.4:** Let \( A \) be the subset of a fine-bi space \( X \) for \( i = 1, 2 \), the fine-bi interior of \( A \) is defined as the union of all \( \tau_i \)-fine-open sets contained in the set \( A \) i.e. the largest \( \tau_i \)-fine-open set contained in the set \( A \) and is denoted by \( \tau_i f_{int}(A) \).

**Definition 3.5:** Let \( A \) be the subset of a fine-bi space \( X \) for \( i = 1, 2 \), the fine-bi closure of \( A \) is defined as the intersection of all \( \tau_f \)-fine-closed sets containing the set \( A \) i.e. the smallest \( \tau_i \)-fine-closed set containing the set \( A \) and is denoted by \( \tau_f f_{cl}(A) \).

**Theorem 3.6:** Let \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) be a fine-bi topological space. Then for \( i = 1, 2 \)
(i) $A$ is $\tau$, fine-open if and only if $A = \tau, f\text{-}\text{int}(A)$.

(ii) $B$ is $\tau$, fine- closed if and only if $B = \tau, f\text{-}\text{cl}(B)$.

**Proof:** Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space and let $A$ and $B$ be any two subsets of $X$.

**Proof of (i):** Let $A$ be a $\tau$, fine-open set.

Then the largest $\tau$, fine-open set containing $A$ is itself.

By the definition 3.4, $\tau, f\text{-}\text{int}(A)$ is the largest $\tau$, fine-open set contained in $A$.

Thus, $A = \tau, f\text{-}\text{int}(A)$.

Conversely, let $A = \tau, f\text{-}\text{int}(A)$.

Since $\tau, f\text{-}\text{int}(A)$ is the largest $\tau$, fine-open set containing $A$ and $A = \tau, f\text{-}\text{int}(A)$, then $A$ is $\tau$, fine-open.

**Proof of (ii):** Let $B$ be a $\tau$, fine-closed set.

Then it is the smallest $\tau$, fine-closed set containing $B$ itself.

By the definition 3.5, $\tau, f\text{-}\text{cl}(B)$ is the smallest $\tau$, fine-closed set containing $B$.

Thus, $B = \tau, f\text{-}\text{cl}(B)$.

Conversely, let $B = \tau, f\text{-}\text{cl}(B)$.

Since $\tau, f\text{-}\text{cl}(B)$ is the smallest $\tau$, fine-closed set containing $B$ and $B = \tau, f\text{-}\text{cl}(B)$ then $B$ is $\tau$, fine-closed.

**Theorem 3.7:** Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. If $\{ A_\alpha : \alpha \in \mathcal{J} \}$ is a collection of subsets of $X$, then for $i = 1, 2$

(i) $\bigcup_{\alpha \in \mathcal{J}} \tau, f\text{-}\text{int}(A_\alpha) \subseteq \tau, f\text{-}\text{int}(\bigcup_{\alpha \in \mathcal{J}} A_\alpha)$

(ii) $\bigcup_{\alpha \in \mathcal{J}} \tau, f\text{-}\text{cl}(A_\alpha) \subseteq \tau, f\text{-}\text{cl}(\bigcup_{\alpha \in \mathcal{J}} A_\alpha)$.

**Proof:** Consider $\{ A_\alpha : \alpha \in \mathcal{J} \}$ is a collection of subsets of $X$.

**Proof of (i):** Let $A_\alpha$ and $A_\beta$ be any two sets of $X$ in this collection of subsets of $X$.

Thus, $A_\alpha \subseteq A_\alpha \cup A_\beta$ and $A_\beta \subseteq A_\alpha \cup A_\beta$.

Then for $i = 1, 2$, $\tau, f\text{-}\text{int}(A_\alpha) \subseteq \tau, f\text{-}\text{int}(A_\alpha \cup A_\beta)$ and $\tau, f\text{-}\text{int}(A_\beta) \subseteq \tau, f\text{-}\text{int}(A_\alpha \cup A_\beta)$.

$\Rightarrow \tau, f\text{-}\text{int}(A_\alpha) \cup \tau, f\text{-}\text{int}(A_\beta) \subseteq \tau, f\text{-}\text{int}(A_\alpha \cup A_\beta)$.

Hence, $\bigcup_{\alpha \in \mathcal{J}} \tau, f\text{-}\text{int}(A_\alpha) \subseteq \tau, f\text{-}\text{int}(\bigcup_{\alpha \in \mathcal{J}} A_\alpha)$, for $i = 1, 2$.

**Proof of (ii):** This proof is same as Proof of (i).

**Theorem 3.8:** Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then arbitrary union of $\tau$, fine-open sets is a $\tau$, fine-open set (for $i = 1, 2$).

**Proof:** Let $\{ G_\alpha : \alpha \in \mathcal{J} \}$ be a collection of $\tau$, fine-open sets of $X$ for $i = 1, 2$.

**Claim:** $\bigcup_{\alpha \in \mathcal{J}} G_\alpha$ is $\tau$, fine-open set (for $i = 1, 2$).

$(\bigcup_{\alpha \in \mathcal{J}} G_\alpha) \cap A_\beta = (G_\alpha \cap A_\beta) \cup (G_\beta \cap A_\beta) \cup ...$

Since $G_\beta \in \tau, f$ (for $i = 1, 2$), there exists an index $\beta \in \mathcal{J}$ such that $G_\beta \cap A_\beta \neq \phi$.

Thus, $(\bigcup_{\alpha \in \mathcal{J}} G_\alpha) \cap A_\beta \neq \phi$.

Hence $\bigcup_{\alpha \in \mathcal{J}} G_\alpha$ is $\tau$, fine-open set (for $i = 1, 2$).
Remark 3.9: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a fine-bitopological space. Then finite intersection of \(\tau_i\), fine-open sets need not be a \(\tau_i\), fine-open set (for \(i = 1, 2\)).

The following example illustrates the above remark.

Example 3.10: Let \(X = \{a, b, c\}\) with the topologies \(\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}\) and \(\tau_2 = \{\phi, X, \{b, c\}\}\). Then \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) is a fine-bitopological space over \(X\), where \(\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}\), \(\tau_2f = \{\phi, X, \{a\}, \{a, b\}, \{b, c\}, \{a, c\}\}\) are \(\tau_i\), fine-open sets (for \(i = 1, 2\)).

Thus, \(\{a, c\}, \{b, c\} \in \tau_1f\) and \(\{a, b\}, \{a, c\} \in \tau_2f\).

But, \(\{a, c\} \cap \{b, c\} = \{c\} \not\in \tau_1f\) and \(\{a, b\} \cap \{a, c\} = \{a\} \not\in \tau_2f\).

Theorem 3.11: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a fine-bitopological space. Then arbitrary intersection of \(\tau_i\), fine-closed sets is a \(\tau_i\), fine-closed set (for \(i = 1, 2\)).

Proof: Let \(\{G_a : \alpha \in J\}\) be a collection of \(\tau_i\), fine-closed sets of \(X\) for \(i = 1, 2\).

Claim: \(\bigcap_{\alpha \in J} G_a\) is a \(\tau_i\), fine-closed set (for \(i = 1, 2\)).

Since \(G_a\) is \(\tau_i\), fine-closed set, \(G_a^c\) is \(\tau_i\), fine-open set (for \(i = 1, 2\)).

By Theorem 3.8, \(\bigcup_{\alpha \in J} G_a^c\) is \(\tau_i\), fine-open set (for \(i = 1, 2\)).

Thus, \((\bigcap_{\alpha \in J} G_a)^c\) is \(\tau_i\), fine-open set (for \(i = 1, 2\)).

Hence \(\bigcap_{\alpha \in J} G_a\) is a \(\tau_i\), fine-closed set (for \(i = 1, 2\)).

Remark 3.12: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a fine-bitopological space. Then finite union of \(\tau_i\), fine-closed sets need not be a \(\tau_i\), fine-closed set (for \(i = 1, 2\)).

The following example illustrates the above remark.

Example 3.13: In Example 3.10, \(\tau_f C = \{\phi, X, \{b, c\}, \{c\}, \{a, c\}, \{a\}\}\), \(\tau_2f C = \{\phi, X, \{a, c\}, \{c\}, \{a\}, \{a, b\}, \{b\}\}\) are \(\tau_i\), fine-closed sets (for \(i = 1, 2\)).

Thus, \(\{a\}, \{b\} \in \tau_f C\) and \(\{b\}, \{c\} \in \tau_2f C\).

But, \(\{a\} \cup \{b\} = \{a, b\} \not\in \tau_f C\) and \(\{b\} \cup \{c\} = \{b, c\} \not\in \tau_2f C\).

Theorem 3.14: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a bi-fine-bitopological space over \(X\). Then \((X, \tau_1, \tau_2, \tau_f)\) and \((X, \tau_2, \tau_2f)\) are fine topological spaces over \(X\).

Proof: The proof of this theorem is a direct-consequence of the definitions.

Theorem 3.15: Every fine topological space over \(X\) is not bi-topological space over \(X\) but is supra bi-topological space and generalized bi-topological space over \(X\).

Proof: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a fine-bitopological space over \(X\).

Then \((X, \tau_1, \tau_f)\) and \((X, \tau_2, \tau_2f)\) are fine topological spaces over \(X\).

Since \(\tau_f\) and \(\tau_2f\) are not topologies on \(X\), \((X, \tau_1, \tau_f)\) and \((X, \tau_2, \tau_2f)\) are not bi-topological spaces over \(X\).

By Theorem 3.8, \((X, \tau_1, \tau_f)\) and \((X, \tau_2, \tau_2f)\) satisfies the axioms of supra bi-topological space and generalized bi-topological space.

Thus, \((X, \tau_1, \tau_f)\) and \((X, \tau_2, \tau_2f)\) are supra bi-topological space and generalized bi-topological space over \(X\).

Definition 3.16: Let \((X, \tau_1, \tau_2, \tau_f, \tau_2f)\) be a fine-bitopological space over \(X\). Then \(A\) is called \(\tau_1, \tau_2f\)-open if \(A \in \tau_1f \cup \tau_2f\), the complement of \(\tau_1, \tau_2f\)-open set is called \(\tau_1, \tau_2f\)-closed set and denoted by \(\tau_1, \tau_2f C\).
**Theorem 3.17:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). Then \(\tau_1 \tau_2 f\)-open set need not necessarily form a topology on \(X\).

**Proof:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). Since \(\tau_1 f\) and \(\tau_2 f\) are not topologies on \(X\), \(\tau_1 \tau_2 f\) is not a topology on \(X\). Thus, \(\tau_1 \tau_2 f\)-open set does not form a topology on \(X\).

**Theorem 3.18:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). Then \((X, \tau_1, \tau_1 f, \tau_2 f)\) and \((X, \tau_2, \tau_1 f, \tau_2 f)\) are not fine topological spaces over \(X\) but are supra bitopological spaces and generalized bi-topological spaces over \(X\).

**Proof:** The proof of this theorem is a direct-consequence of the definitions.

**Theorem 3.19:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). If \(\tau_1 \tau_2 f\) contains all subsets of \(X\), then \((X, \tau_1, \tau_1 f, \tau_2 f)\) and \((X, \tau_2, \tau_1 f, \tau_2 f)\) are bi-topological spaces over \(X\).

**Proof:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). If \(\tau_1 \tau_2 f\) contains all subsets of \(X\), then \(\tau_1 \tau_2 f\) is a discrete topology over \(X\). Hence, \((X, \tau_1, \tau_1 \tau_2 f)\) and \((X, \tau_2, \tau_1 \tau_2 f)\) are bi-topological spaces over \(X\).

**Theorem 3.20:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space over \(X\). Then \((X, \tau_1 \tau_2, \tau_1 f, \tau_2 f)\) is a fine topological space over \(X\).

**Proof:** The proof of this theorem is a direct-consequence of the definitions.

**Definition 3.21:** Let \(A\) be the subset of a fine-bi space \(X\), the bi-fine interior of \(A\) is defined as the union of all \(\tau_1 \tau_2\) fine-open sets contained in the set \(A\) i.e. the largest \(\tau_1 \tau_2\) fine-open set contained in the set \(A\) and is denoted by \(\tau_1 \tau_2 f\text{-int}(A)\).

**Definition 3.22:** Let \(A\) be the subset of a fine-bi space \(X\), the bi-fine closure of \(A\) is defined as the intersection of all \(\tau_1 \tau_2\) fine-closed sets containing the set \(A\) i.e. the smallest \(\tau_1 \tau_2\) fine-closed set containing the set \(A\) and is denoted by \(\tau_1 \tau_2 f\text{-cl}(A)\).

**Theorem 3.23:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space and \(A\) be any subset of \(X\). Then the following are true:

(i) \(\tau_1 \tau_2 -\text{int}(A) \subseteq \tau_1 \tau_2 \text{-int}(A)\)

(ii) \(\tau_1 \text{-int}(A) \subseteq \tau_1 \tau_2 \text{-int}(A)\)

(iii) \(\tau_2 \text{-int}(A) \subseteq \tau_1 \tau_2 \text{-int}(A)\)

**Proof:** The proof of this theorem is a direct-consequence of the definitions.

**Theorem 3.24:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space and \(A\) be any subset of \(X\). Then the following are true:

(i) \(\tau_1 \tau_2 -\text{cl}(A) \supseteq \tau_1 \tau_2 \text{-cl}(A)\)

(ii) \(\tau_1 \text{-cl}(A) \supseteq \tau_1 \tau_2 \text{-cl}(A)\)

(iii) \(\tau_2 \text{-cl}(A) \supseteq \tau_1 \tau_2 \text{-cl}(A)\)

**Proof:** The proof of this theorem is a direct-consequence of the definitions.

**Theorem 3.25:** Let \((X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)\) be a fine-bitopological space. Then arbitrary union of \(\tau_1 \tau_2\) fine-open sets is a \(\tau_1 \tau_2\) fine-open set and arbitrary intersection of \(\tau_1 \tau_2\) fine-closed sets is a \(\tau_1 \tau_2\) fine-closed set.

**Proof:** This proof is same as Theorem 3.8 and Theorem 3.11.
Remark 3.26: Let \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) be a fine-bitopological space. Then finite intersection of \(\tau_1, \tau_2\) fine-open sets need not be a \(\tau_1, \tau_2\) fine-open set and finite union of \(\tau_1, \tau_2\) fine-closed sets need not be a \(\tau_1, \tau_2\) fine-closed set.

The following example illustrates the above remark.

Example 3.27: Let \(X = \{a, b, c\}\) with the topologies \(\tau_1 = \{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X, \{b\}\}\). Then \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) is a fine-bitopological space over \(X\), where 
\[
\tau_1f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\},
\]
\[
\tau_2f = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}\text{ are } \tau_i\text{ fine-open sets (for } i = 1, 2)\text{.}
\]

Then \(\tau_1f \cap \tau_2f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}\) is \(\tau_1f\) fine-open set and \(\tau_1f \cup \tau_2f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) is \(\tau_1f\) fine-closed set.

Thus, \(\tau_1f \cap \tau_2f\) then \(\tau_1f \cup \tau_2f\) is \(\tau_1f\) fine-open set and denoted by \(\tau_1f\) C.

Remark 3.28: Let \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) be a fine-bitopological space over \(X\). Then \(A\) is called \(\tau_1f\) open if \(A = A_1 \cup B_1\), where \(A_1 \in \tau_1f, B_1 \in \tau_2f\), the complement of \(\tau_1f\) open set is called \(\tau_1f\) closed set and denoted by \(\tau_1fC\).

Example 3.30: Let \(X = \{a, b, c\}\) with the topologies \(\tau_1 = \{\phi, X, \{a\}\}\) and \(\tau_2 = \{\phi, X, \{c\}\}\). Then \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) is a fine-bitopological space over \(X\), where 
\[
\tau_1f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\},
\]
\[
\tau_2f = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}\text{ are } \tau_i\text{ fine-open sets (for } i = 1, 2)\text{.}
\]

Then \(\tau_1f = \{\phi, X, \{a\}, \{a, c\}\}\) and \(\tau_1f = \{\phi, X, \{a\}, \{a, c\}\}\). Also \(\tau_1f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}\) and \(\tau_1f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}\). Thus, \(\tau_1f\) open set is not equal to \(\tau_1f\) open set, but \(\tau_1f\) open set is equal to \(\tau_1f\) open set over \(X\).

4. \(\tau_1, \tau_2, fb\)-Open Sets:

In this section \(X\) denotes the fine-bitopological space \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) and \(\tau_1, \tau_2\) \(fb\)-open sets in fine-bitopological spaces have been defined.

Definition 4.1: A subset \(A\) of \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) is said to be \(\tau_1, \tau_2, fb\)-open set if 
\[
A \subseteq \tau_1f, \text{int}(\tau_1 \text{f}, \text{cl}(A)) \cup \tau_2f, \text{cl}(\tau_2 \text{f}, \text{int}(A)).
\]

The family of all \(\tau_1, \tau_2, fb\)-open set is denoted by \(\tau_1, \tau_2, fb\)-O.

Definition 4.2: The complement of \(\tau_1, \tau_2, fb\)-open set is said to be \(\tau_1, \tau_2, fb\)-closed set and is denoted by \(\tau_1, \tau_2, fb\)-C.

Definition 4.3: A subset \(A\) of \((X, \tau_1, \tau_2, \tau_1f, \tau_2f)\) is said to be \(\tau_2, \tau_1, fb\)-open set if 
\[
A \subseteq \tau_2f, \text{int}(\tau_2f, \text{cl}(A)) \cup \tau_1f, \text{cl}(\tau_1f, \text{int}(A)).
\]

The family of all \(\tau_2, \tau_1, fb\)-open set is denoted by \(\tau_2, \tau_1, fb\)-O.

Definition 4.4: The complement of \(\tau_2, \tau_1f, fb\)-open set is said to be \(\tau_2, \tau_1, fb\)-closed set and is denoted by \(\tau_2, \tau_1, fb\)-C.
**Definition 4.5:** A subset $A$ of a fine-bitopological space $(X, \tau_1, \tau_2, \tau_f, \tau_{if})$ is called pairwise $fb$-open in $(X, \tau_1, \tau_2, \tau_f, \tau_{if})$ if $A$ is $\tau_1, \tau_2, \tau_f$-$fb$-open and $\tau_{if}$-$fb$-open.

**Definition 4.6:** A subset $A$ of $X$ for $i, j = 1, 2$ and $i \neq j$, is said to be $\tau_i, \tau_j$-$fb$-interior of $A$ if the union of all $\tau_i, \tau_j$-$fb$-open sets contained in $A$. The $\tau_i, \tau_j$-$fb$-interior of $A$ is denoted by $\tau_i, \tau_j$-$fb$-$int(A)$.

**Definition 4.7:** A subset $A$ of $X$ for $i, j = 1, 2$ and $i \neq j$, is said to be $\tau_i, \tau_j$-$fb$-closure of $A$ if the intersection of all $\tau_i, \tau_j$-$fb$-open sets containing $A$. The $\tau_i, \tau_j$-$fb$-closure of $A$ is denoted by $\tau_i, \tau_j$-$fb$-$cl(A)$.

**Example 4.8:** Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Then $(X, \tau_1, \tau_2, \tau_f, \tau_{if})$ is the bi-fine topological space over $X$, where $\tau_f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\tau_{if} = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$. Thus $\tau_1, \tau_f$-$O = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$, $\tau_1, \tau_{if}$-$O = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b, c\}\}$.

Then pair wise $fb$-open sets are $\phi, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$.

**Theorem 4.9:** Arbitrary union of $\tau_1, \tau_2$-fine-$b$-open sets is a $\tau_1, \tau_2$-fine-$b$-open set in $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$. 

**Proof:** Let $\{A_\alpha : \alpha \in J\}$ be a collection of $\tau_1, \tau_2$-fine-$b$-open sets of $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$. Then $A_\alpha \subseteq \tau_1, \tau_{if}$-$int(\tau_{if}f$-$cl(A_\alpha)) \cup \tau_{if}f$-$cl(\tau_1, \tau_{if}$-$int(\tau_{if}f$-$cl(A_\alpha)$)

By Remark 3.7, $\bigcup_{\alpha \in J} A_\alpha \subseteq \tau_1, \tau_{if}$-$int(\tau_{if}f$-$cl(\bigcup_{\alpha \in J} A_\alpha) \cup \tau_{if}f$-$cl(\tau_1, \tau_{if}$-$int(\bigcup_{\alpha \in J} A_\alpha)$)

**Theorem 4.10:** Arbitrary union of $\tau_2, \tau_1$-fine-$b$-open sets is a $\tau_2, \tau_1$-fine-$b$-open set in $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$. 

**Proof:** The proof is same as Theorem 4.9.

**Remark 4.11:** Finite intersection of $\tau_1, \tau_2$-fine-$b$-open sets need not be a $\tau_1, \tau_2$-fine-$b$-open set (for $i, j = 1, 2$ and $i \neq j$) in $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$.

The following example illustrates the above remark.

**Example 4.12:** In Example 4.8, $\{a, c\}, \{b, c\} \in \tau_1, \tau_2$-$fb$-$O$ and $\{a, b\}, \{a, c\} \in \tau_2, \tau_1$-$fb$-$O$.

But, $\{a, c\} \cap \{b, c\} = \{c\} \notin \tau_1, \tau_2$-$fb$-$O$ and $\{a, b\} \cap \{a, c\} = \{a\} \notin \tau_2, \tau_1$-$fb$-$O$.

**Theorem 4.13:** Every $\tau_1, \tau_2$-fine-$b$-open set is $\tau_1$-fine-open set in $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$. 

**Proof:** The proof of this theorem is straightforward.

**Theorem 4.14:** Every $\tau_2, \tau_1$-fine-$b$-open set is $\tau_2$-fine-open set in $(X, \tau_1, \tau_2, \tau_{if}, \tau_{if}f)$. 

**Proof:** The proof of this theorem is straightforward.

**References:**