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$\tau_1\tau_2$-*FB*-OPEN SETS IN FINE - BITOPOLOGICAL SPACES

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Abstract:

In this paper, fine-bitopological spaces and a new class of open sets in it are defined. Also some of their basic properties are discussed.

Index Terms: τ_i fine-open sets (for *i* = 1, 2), $\tau_1\tau_2$ fine-open sets, $\tau_{1,2}$ fine-open sets, $\tau_1\tau_2$ *fb*-open sets & $\tau_2\tau_1$ *fb*-open sets.

1. Introduction:

Topology plays a key role in many fields of information systems, particle physics, quantum physics and high energy physics and several branches of mathematics such as data mixing, computational topology for geometric design, computer aided geometric design and digital topology. In 1963, Kelly [4] introduced the notion of bitopological spaces; the ordered triple (X, τ_1, τ_2) where τ_1 and τ_2 are topologies on *X*, is called bitopological space. Andrijev [1] introduced the concept of *b*-open sets in topological spaces in 1996. Later Abo Khadra and Nasef discussed *b*-open sets in bitopological spaces in 2003. Many researchers studied characterization of bitpological spaces [2, 3]. In 2012, Powar and Rajak [5] have investigated a special case of generalized topological space called fine topological space.

2. Preliminaries:

Throughout the paper (X, τ) and (X, τ_1, τ_2) denote topological space and bitopological space without separation axioms unless specified. Let $A \subseteq X$. The closure and the interior of a subset *A* of a bitopological space (X, τ_1, τ_2) with respective to τ_i (*i* = 1, 2) will be denoted by τ_i-cl(*A*) and τ_i-int(*A*). By (*i, j*) we mean the pair of topologies (τ_i, τ_j), for *i, j* = 1, 2 and $i \neq j$.

Definition 2.1 [3]: A subset *A* of a bitopological space (X, τ_1, τ_2) is said to be

(i) (*i, j*)-*b*-open if $A \subset \tau_i$-int(τ_j-cl(*A*)) $\cup$ τ_j-cl(τ_i-int(*A*)).

(ii) (*i, j*)-*b*-closed if $A \subset \tau_i$-int(τ_j-cl(*A*)) $\cap$ τ_j-cl(τ_i-int(*A*)).

Definition 2.2 [3]: Let *A* be a subset of a bitopological space (X, τ_1, τ_2).

(i) The (*i, j*)-*b*-closure of *A* denoted by (*i, j*)-*b*-cl(*A*), is defined by the intersection of all (*i, j*)-*b*-closed sets containing *A*.

(ii) The (*i, j*)-*b*-interior of *A* denoted by (*i, j*)-*b*-int(*A*), is defined by the union of all (*i, j*)-*b*-open sets contained in *A*.

Definition 2.3 [2]: *A* is called $\tau_1\tau_2$-open if $A \in \tau_1 \cup \tau_2$, the complement of $\tau_1\tau_2$-open set is called $\tau_1\tau_2$-closed set.

Definition 2.4 [2]: *A* is called $\tau_{1,2}$-open if $A = A_i \cup B_i$, where $A_i \in \tau_1$, $B_i \in \tau_2$, the complement of $\tau_{1,2}$-open set is called $\tau_{1,2}$-closed set.

Remark 2.5 [2]: Note that $\tau_1\tau_2$-open set of (X, τ_1, τ_2) need not necessarily form a topology on (X, τ_1, τ_2).

Definition 2.6 [5]: Let (X, τ) be a topological space we define

$\tau(A_\alpha) = \tau_\alpha$ (say) = $\{G_\alpha (\neq X) : G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \tau$ and $A_\alpha \neq \phi, X$, for some $\alpha \in J$, where J is the index set$\}$.

Now, we define $\tau_f = \{\phi, X, \bigcup_{\alpha \in J} \{\tau_\alpha\}\}$

The above collection τ_f of subsets of X is called the fine collection of subsets of X and (X, τ, τ_f) is said to be the fine space X generated by the topology τ on X.

Definition 2.7 [5]: A subset U of a fine space X is said to be fine-open sets of X, if U belongs to the collection τ_f and the complement of every fine-open sets of X is called the fine-closed sets of X and we denote the collection by F_f.

Definition 2.8 [5]: Let A be the subset of a fine space X, the fine interior of A is defined as the union of all fine-open sets contained in the set A i.e. the largest fine-open set contained in the set A and is denoted by f_{int}.

Definition 2.9 [5]: Let A be the subset of a fine space X, the fine closure of A is defined as the intersection of all fine-closed sets containing the set A i.e. the smallest fine-closed set containing the set A and is denoted by f_{cl}.

Remark 2.10 [5]: Let (X, τ, τ_f) be a fine space and A be any arbitrary subset of X. Then:

(1) int $(A) \subseteq f_{\text{int}}(A)$

(2) $f_{cl}(A) \subseteq$ cl (A)

3. Fine-Bitopological Spaces:

In this section X denotes the fine-bitopological space $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Definition 3.1: Let (X, τ_1, τ_2) be a bi-topological space we define

$\tau_i(A_\alpha) = \tau_{i\alpha}$ (say) = $\{G_\alpha (\neq X) : G_\alpha \subset X, G_\alpha \cap A_\alpha \neq \phi$, for $A_\alpha \in \tau_i$ and $A_\alpha \neq \phi, X$, for some $\alpha \in J$, where J is the index set and for $i = 1, 2\}$.

Now, we define $\tau_i f = \{\phi, X, \bigcup_{\alpha \in J} \{\tau_{i\alpha}\}\}$, for $i = 1, 2$.

The above collection $\tau_i f$ ($i = 1, 2$) of subsets of X is called the fine-bi collection of subsets of X and the quintuple $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is said to be the fine-bi space X generated by the topologies τ_1 and τ_2 on X.

Definition 3.2: A subset U of a fine-bi space X is said to be τ_i fine-open set of X for $i = 1, 2$, if U belongs to the collection $\tau_i f$.

Definition 3.3: The complement of every τ_i fine-open set of X is called the τ_i fine-closed set of X for $i = 1, 2$ and this collection is denoted by $\tau_i f$C.

Definition 3.4: Let A be the subset of a fine-bi space X for $i = 1, 2$, the fine-bi interior of A is defined as the union of all τ_i fine-open sets contained in the set A i.e. the largest τ_i fine-open set contained in the set A and is denoted by $\tau_i f$-int(A).

Definition 3.5: Let A be the subset of a fine-bi space X for $i = 1, 2$, the fine-bi closure of A is defined as the intersection of all τ_i fine-closed sets containing the set A i.e. the smallest τ_i fine-closed set containing the set A and is denoted by $\tau_i f$-cl(A).

Theorem 3.6: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then for $i = 1, 2$

(i) A is τ_i fine-open if and only if A = $\tau_i f$-int(A).

(ii) B is τ_i fine- closed if and only if B = $\tau_i f$-cl(B).

Proof: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space and let A and B be any two subsets of X.

Proof of (i): Let A be a τ_i fine-open set.

Then the largest τ_i fine-open set containing A is itself.

By the definition 3.4, $\tau_i f$-int(A) is the largest τ_i fine-open set contained in A.

Thus, A = $\tau_i f$-int(A).

Conversely, let A = $\tau_i f$-int(A).

Since $\tau_i f$-int(A) is the largest τ_i fine-open set containing A and A = $\tau_i f$-int(A), then A is τ_i fine-open.

Proof of (ii): Let B be a τ_i fine-closed set.

Then it is the smallest τ_i fine-closed set containing B itself.

By the definition 3.5, $\tau_i f$-cl(B) is the smallest τ_i fine-closed set containing B.

Thus, B = $\tau_i f$-cl(B).

Conversely, let B = $\tau_i f$-cl(B).

Since $\tau_i f$-cl(B) is the smallest τ_i fine-closed set containing B and B = $\tau_i f$-cl(B) then B is τ_i fine-closed.

Theorem 3.7: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. If $\{A_\alpha : \alpha \in J\}$ is a collection of subsets of X, then for i = 1, 2

(i) $\bigcup_{\alpha\in J} \tau_i f$-int$(A_\alpha) \subseteq \tau_i f$-int$(\bigcup_{\alpha\in J} A_\alpha)$

(ii) $\bigcup_{\alpha\in J} \tau_i f$-cl$(A_\alpha) \subseteq \tau_i f$-cl$(\bigcup_{\alpha\in J} A_\alpha)$.

Proof: Consider $\{A_\alpha : \alpha \in J\}$ is a collection of subsets of X.

Proof of (i): Let A_α and A_β be any two sets of X in this collection of subsets of X.

Thus, $A_\alpha \subseteq A_\alpha \cup A_\beta$ and $A_\beta \subseteq A_\alpha \cup A_\beta$.

Then for i = 1, 2, $\tau_i f$-int$(A_\alpha) \subseteq \tau_i f$-int$(A_\alpha \cup A_\beta)$ and $\tau_i f$-int$(A_\beta) \subseteq \tau_i f$-int$(A_\alpha \cup A_\beta)$.

$\Rightarrow \tau_i f$-int$(A_\alpha) \cup \tau_i f$-int$(A_\beta) \subseteq \tau_i f$-int$(A_\alpha \cup A_\beta)$.

Hence, $\bigcup_{\alpha\in J} \tau_i f$-int$(A_\alpha) \subseteq \tau_i f$-int$(\bigcup_{\alpha\in J} A_\alpha)$, for i = 1, 2.

Proof of (ii): This proof is same as Proof of (i).

Theorem 3.8: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then arbitrary union of τ_i fine-open sets is a τ_i fine-open set (for i = 1, 2).

Proof: Let $\{G_\alpha : \alpha \in J\}$ be a collection of τ_i fine-open sets of X for i = 1, 2.

Claim: $\bigcup_{\alpha\in J} G_\alpha$ is τ_i fine-open set (for i = 1, 2).

$(\bigcup_{\alpha\in J} G_\alpha) \cap A_\beta = (G_\alpha \cap A_\beta) \cup (G_\beta \cap A_\beta) \cup \ldots$

Since $G_\beta \in \tau_i f$ (for i = 1, 2), there exists an index $\beta \in J$ such that $G_\beta \cap A_\beta \neq \phi$.

Thus, $(\bigcup_{\alpha\in J} G_\alpha) \cap A_\beta \neq \phi$.

Hence $\bigcup_{\alpha\in J} G_\alpha$ is τ_i fine-open set (for i = 1, 2).

Remark 3.9: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then finite intersection of τ_i fine-open sets need not be a τ_i fine-open set (for i = 1, 2).

The following example illustrates the above remark.

Example 3.10: Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Then $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is a fine-bitopological space over X, where

$\tau_1 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}$,

$\tau_2 f = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{c\}, \{a, c\}\}$ are τ_i fine-open sets (for i =1, 2).

Thus, $\{a, c\}, \{b, c\} \in \tau_1 f$ and $\{a, b\}, \{a, c\} \in \tau_2 f$.

But, $\{a, c\} \cap \{b, c\} = \{c\} \notin \tau_1 f$ and $\{a, b\} \cap \{a, c\} = \{a\} \notin \tau_2 f$.

Theorem 3.11: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then arbitrary intersection of τ_i fine-closed sets is a τ_i fine-closed set (for i = 1, 2).

Proof: Let $\{G_\alpha : \alpha \in J\}$ be a collection of τ_i fine- closed sets of X for i = 1, 2.

Claim: $\bigcap_{\alpha \in J} G_\alpha$ is τ_i fine-closed set (for i = 1, 2).

Since G_α is τ_i fine- closed set, G_α^c is τ_i fine- open set (for i = 1, 2).

By Theorem 3.8, $\bigcup_{\alpha \in J} G_\alpha^c$ is τ_i fine-open set (for i = 1, 2).

Thus, $(\bigcap_{\alpha \in J} G_\alpha)^c$ is τ_i fine-open set (for i = 1, 2).

Hence $\bigcap_{\alpha \in J} G_\alpha$ is τ_i fine-closed set (for i = 1, 2).

Remark 3.12: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then finite union of τ_i fine-closed sets need not be a τ_i fine-closed set (for i = 1, 2).

The following example illustrates the above remark.

Example 3.13: In Example 3.10, $\tau_1 f\text{C} = \{\phi, X, \{b, c\}, \{c\}, \{b\}, \{a, c\}, \{a\}\}$,

$\tau_2 f\text{C} = \{\phi, X, \{a, c\}, \{c\}, \{a\}, \{a, b\}, \{b\}\}$ are τ_i fine-closed sets (for i = 1, 2).

Thus, $\{a\}, \{b\} \in \tau_1 f\text{C}$ and $\{b\}, \{c\} \in \tau_2 f\text{C}$.

But, $\{a\} \cup \{b\} = \{a, b\} \notin \tau_1 f\text{C}$ and $\{b\} \cup \{c\} = \{b, c\} \notin \tau_2 f\text{C}$.

Theorem 3.14: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a bi-fine-bitopological space over X. Then $(X, \tau_1, \tau_1 f)$ and $(X, \tau_2, \tau_2 f)$ are fine topological spaces over X.

Proof: The proof of this theorem is a direct-consequence of the definitions.

Theorem 3.15: Every fine topological space over X is not bi-topological space over X but is supra bi-topological space and generalized bi-topological space over X.

Proof: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X.

Then $(X, \tau_1, \tau_1 f)$ and $(X, \tau_2, \tau_2 f)$ are fine topological spaces over X.

Since $\tau_1 f$ and $\tau_2 f$ are not topologies on X, $(X, \tau_1, \tau_1 f)$ and $(X, \tau_2, \tau_2 f)$ are not bi-topological spaces over X.

By Theorem 3.8, $(X, \tau_1, \tau_1 f)$ and $(X, \tau_2, \tau_2 f)$ satisfies the axioms of supra bi-topological space and generalized bi-topological space.

Thus, $(X, \tau_1, \tau_1 f)$ and $(X, \tau_2, \tau_2 f)$ are supra bi-topological space and generalized bi-topological space over X.

Definition 3.16: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then A is called $\tau_1\tau_2 f$-open if $A \in \tau_1 f \cup \tau_2 f$, the complement of $\tau_1\tau_2 f$-open set is called $\tau_1\tau_2 f$-closed set and denoted by $\tau_1\tau_2 f\text{C}$.

Theorem 3.17: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then $\tau_1\tau_2 f$-open set need not necessarily form a topology on X.

Proof: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X.

Since $\tau_1 f$ and $\tau_2 f$ are not topologies on X, $\tau_1\tau_2 f$ is not a topology on X.

Thus, $\tau_1\tau_2 f$-open set does not form a topology on X.

Theorem 3.18: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then $(X, \tau_1, \tau_1\tau_2 f)$ and $(X, \tau_2, \tau_1\tau_2 f)$ are not fine topological spaces over X but are supra bi-topological spaces and generalized bi-topological spaces over X.

Proof: The proof of this theorem is a direct-consequence of the definitions.

Theorem 3.19: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. If $\tau_1\tau_2 f$ contains all subsets of X, then $(X, \tau_1, \tau_1\tau_2 f)$ and $(X, \tau_2, \tau_1\tau_2 f)$ are bi-topological spaces over X.

Proof: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X.

If $\tau_1\tau_2 f$ contains all subsets of X, then $\tau_1\tau_2 f$ is a discrete topology over X.

Hence, $(X, \tau_1, \tau_1\tau_2 f)$ and $(X, \tau_2, \tau_1\tau_2 f)$ are bi-topological spaces over X.

Theorem 3.20: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then $(X, \tau_1\tau_2, \tau_1\tau_2 f)$ is a fine topological space over X.

Proof: The proof of this theorem is a direct-consequence of the definitions.

Definition 3.21: Let A be the subset of a fine-bi space X, the bi-fine interior of A is defined as the union of all $\tau_1\tau_2$ fine-open sets contained in the set A i.e. the largest $\tau_1\tau_2$ fine-open set contained in the set A and is denoted by $\tau_1\tau_2 f$-int(A).

Definition 3.22: Let A be the subset of a fine-bi space X, the bi-fine closure of A is defined as the intersection of all $\tau_1\tau_2$ fine-closed sets containing the set A i.e. the smallest $\tau_1\tau_2$ fine-closed set containing the set A and is denoted by $\tau_1\tau_2 f$-cl(A).

Theorem 3.23: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space and A be any subset of X. Then the following are true:

(i) $\tau_1\tau_2$-int$(A) \subseteq \tau_1\tau_2 f$-int (A)

(ii) τ_1-int$(A) \subseteq \tau_1\tau_2 f$-int (A)

(iii) τ_2-int$(A) \subseteq \tau_1\tau_2 f$-int (A)

Proof: The proof of this theorem is a direct-consequence of the definitions.

Theorem 3.24: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space and A be any subset of X. Then the following are true:

(i) $\tau_1\tau_2$-cl$(A) \supseteq \tau_1\tau_2 f$-cl$(A)$

(ii) τ_1-cl$(A) \supseteq \tau_1\tau_2 f$-cl$(A)$

(iii) τ_2-cl$(A) \supseteq \tau_1\tau_2 f$-cl$(A)$

Proof: The proof of this theorem is a direct-consequence of the definitions.

Theorem 3.25: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then arbitrary union of $\tau_1\tau_2$ fine-open sets is a $\tau_1\tau_2$ fine-open set and arbitrary intersection of $\tau_1\tau_2$ fine-closed sets is a $\tau_1\tau_2$ fine- closed set.

Proof: This proof is same as Theorem 3.8 and Theorem 3.11.

Remark 3.26: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space. Then finite intersection of $\tau_1\tau_2$ fine-open sets need not be a $\tau_1\tau_2$ fine-open set and finite union of $\tau_1\tau_2$ fine- closed sets need not be a $\tau_1\tau_2$ fine- closed set.

The following example illustrates the above remark.

Example 3.27: Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{b\}\}$. Then $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is a fine-bitopological space over X, where

$\tau_1 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$,

$\tau_2 f = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}\}$ are τ_i fine-open sets (for $i = 1, 2$).

Then $\tau_1\tau_2 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{c\}, \{b, c\}\}$ is $\tau_1\tau_2$ fine-open set and

$\tau_1\tau_2 f\text{C} = \{\phi, X, \{b, c\}, \{c\}, \{b\}, \{a, b\}, \{a\}\}$ is $\tau_1\tau_2$ fine-closed set.

Thus, $\{a, b\}, \{b, c\} \in \tau_1\tau_2 f$, then $\{a, b\} \cap \{b, c\} = \{b\} \notin \tau_1\tau_2 f$.

Also, $\{a\}, \{c\} \in \tau_1\tau_2 f\text{C}$, then $\{a\} \cup \{c\} = \{a, c\} \notin \tau_1\tau_2 f\text{C}$.

Definition 3.28: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then A is called $\tau_{1,2} f$-open if $A = A_i \cup B_i$, where $A_i \in \tau_1 f$, $B_i \in \tau_2 f$, the complement of $\tau_{1,2} f$-open set is called $\tau_{1,2} f$-closed set and denoted by $\tau_{1,2} f\text{C}$.

Remark 3.29: Let $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ be a fine-bitopological space over X. Then $\tau_1\tau_2$-open set need not necessarily equal to $\tau_{1,2}$-open set, but $\tau_1\tau_2 f$-open set is equal to $\tau_{1,2} f$-open set over X.

The following example illustrates the above remark.

Example 3.30: Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}\}$ and $\tau_2 = \{\phi, X, \{c\}\}$. Then $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is a fine-bitopological space over X, where

$\tau_1 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}\}$,

$\tau_2 f = \{\phi, X, \{c\}, \{a, c\}, \{b, c\}\}$ are τ_i fine-open sets (for $i = 1, 2$).

Then $\tau_1\tau_2 = \{\phi, X, \{a\}, \{c\}\}$ and $\tau_{1,2} = \{\phi, X, \{a\}, \{a, c\}, \{c\}\}$

Also $\tau_1\tau_2 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{c\}, \{b, c\}\}$ and $\tau_{1,2} f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{c\}, \{b, c\}\}$. Thus, $\tau_1\tau_2$-open set is not equal to $\tau_{1,2}$-open set, but $\tau_1\tau_2 f$-open set is equal to $\tau_{1,2} f$-open set over X.

4. $\tau_1\tau_2$ *fb*-Open Sets:

In this section X denotes the fine-bitopological space $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ and $\tau_1\tau_2$ *fb*-open sets in fine-bitopological spaces have been defined.

Definition 4.1: A subset A of $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is said to be $\tau_1\tau_2 fb$-open set if $A \subseteq \tau_1 f\text{-int}(\tau_2 f\text{-cl}(A)) \cup \tau_2 f\text{-cl}(\tau_1 f\text{-int}(A))$.

The family of all $\tau_1\tau_2 fb$-open set is denoted by $\tau_1\tau_2 fb$-O.

Definition 4.2: The complement of $\tau_1\tau_2 fb$-open set is said to be $\tau_1\tau_2 fb$-closed set and is denoted by $\tau_1\tau_2 fb$-C.

Definition 4.3: A subset A of $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is said to be $\tau_2\tau_1 fb$-open set if $A \subseteq \tau_2 f\text{-int}(\tau_1 f\text{-cl}(A)) \cup \tau_1 f\text{-cl}(\tau_2 f\text{-int}(A))$.

The family of all $\tau_2\tau_1 fb$-open set is denoted by $\tau_2\tau_1 fb$-O.

Definition 4.4: The complement of $\tau_2\tau_1 fb$-open set is said to be $\tau_2\tau_1 fb$-closed set and is denoted by $\tau_2\tau_1 fb$-C.

Definition 4.5: A subset A of a fine-bitopological space $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is called pair wise *fb*-open in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ if A is $\tau_1\tau_2 fb$-open and $\tau_2\tau_1 fb$-open.

Definition 4.6: A subset A of X for *i, j* = 1, 2 and $i \neq j$, is said to be $\tau_i\tau_j fb$-interior of A if the union of all $\tau_i\tau_j fb$-open sets contained in A. The $\tau_i\tau_j fb$-interior of A is denoted by $\tau_i\tau_j fb$-int(A).

Definition 4.7: A subset A of X for *i, j* = 1, 2 and $i \neq j$, is said to be $\tau_i\tau_j fb$-closure of A if the intersection of all $\tau_i\tau_j fb$-open sets containing A. The $\tau_i\tau_j fb$-closure of A is denoted by $\tau_i\tau_j fb$-cl(A).

Example 4.8: Let $X = \{a, b, c\}$ with the topologies $\tau_1 = \{\phi, X, \{a\}, \{a, b\}\}$ and $\tau_2 = \{\phi, X, \{b, c\}\}$. Then $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$ is the bi-fine topological space over X, where

$\tau_1 f = \{\phi, X, \{a\}, \{a, b\}, \{a, c\}, \{b\}, \{b, c\}\}$, $\tau_2 f = \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{c\}, \{a, c\}\}$.

Thus $\tau_1\tau_2 fb$-O $= \{\phi, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}\}$,

$\tau_2\tau_1 fb$-O $= \{\phi, X, \{b\}, \{a, b\}, \{b, c\}, \{c\}, \{a, c\}\}$.

Then pair wise *fb*-open sets are $\phi, X, \{b\}, \{a, b\}, \{a, c\}, \{b, c\}$.

Theorem 4.9: Arbitrary union of $\tau_1\tau_2$ fine-*b*-open sets is a $\tau_1\tau_2$ fine-*b*-open set in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Proof: Let $\{A_\alpha : \alpha \in J\}$ be a collection of $\tau_1\tau_2$ fine-*b*-open sets of $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Then $A_\alpha \subseteq \tau_1 f$-int($\tau_2 f$-cl($A_\alpha$)) $\cup$ $\tau_2 f$-cl($\tau_1 f$-int(A_α)).

By Remark 3.7, $\bigcup_{\alpha \in J} A_\alpha \subseteq \tau_1 f$-int($\tau_2 f$-cl($\bigcup_{\alpha \in J} A_\alpha$) $\cup$ $\tau_2 f$-cl($\tau_1 f$-int($\bigcup_{\alpha \in J} A_\alpha$).

Theorem 4.10: Arbitrary union of $\tau_2\tau_1$ fine-*b*-open sets is a $\tau_2\tau_1$ fine-*b*-open set in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Proof: The proof is same as Theorem 4.9.

Remark 4.11: Finite intersection of $\tau_i\tau_j$ fine-*b*-open sets need not be a $\tau_i\tau_j$ fine-*b*-open set (for *i, j* = 1, 2 and $i \neq j$) in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

The following example illustrates the above remark.

Example 4.12: In Example 4.8, $\{a, c\}, \{b, c\} \in \tau_1\tau_2 fb$-O and $\{a, b\}, \{a, c\} \in \tau_2\tau_1 fb$-O.

But, $\{a, c\} \cap \{b, c\} = \{c\} \notin \tau_1\tau_2 fb$-O and $\{a, b\} \cap \{a, c\} = \{a\} \notin \tau_2\tau_1 fb$-O.

Theorem 4.13: Every $\tau_1\tau_2$ fine-*b*-open set is τ_1 fine-open set in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Proof: The proof of this theorem is straight forward.

Theorem 4.14: Every $\tau_2\tau_1$ fine-*b*-open set is τ_2 fine-open set in $(X, \tau_1, \tau_2, \tau_1 f, \tau_2 f)$.

Proof: The proof of this theorem is straight forward.

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