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PRODUCT OPERATION ON FUZZY TRANSITION MATRICES

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Abstract:

Fuzzy Matrices over fuzzy algebra plays a significant role in matrix theory over fuzzy settings. Fuzzy transition matrix is another interesting structure in fuzzy matrix theory. Our aim in this paper is to define product operation on fuzzy transition matrix and to provide some interesting result on it.

1. Introduction:

Fuzzy set theory was proposed by Lotifi A. Zadeh [11] and it has extensive applications in various fields. Fuzzy matrix was introduced by Thomason [7] and the concept of uncertainty was discussed by using fuzzy matrices. Different concepts and ideas of fuzzy matrices have been given earlier mainly by Kim, Meenakshi and Thomson [1, 2, 7]. Fuzzy matrix plays a vital role in fuzzy modeling, fuzzy diagnosis and fuzzy controls. It also has applications in fields like psychology, medicine, economics and sociology. It is well known that the matrix formulation gives an extraordinary advantage to handle any data or mathematical problems under consideration. In other words, when some mathematical problems are not solved by the classical matrices, then the concept of fuzzy matrix is used. It is sure that fuzzy matrices have advantageous applications over fuzzy algebra and give decision making results in an effective manner.

A Stochastic process is a mathematical model that evolves over time in a probabilistic manner. A special kind of stochastic process is called Markov chain. Transition probability matrix plays an important role in Markov process. Its applications in medicine are quite common and have become a standard tool in decision making [3, 4, 5, 6].

In the early investigation, Vijayabalaji et.al. studied the algebraic tool structure in fuzzy matrix and furnished new definition namely fuzzy transition matrix [8] and provided some result on it. In their another findings, they introduced orthogonal cubic transition matrices using inner product [10]. Further, they also discussed various operations of interval valued intuitionistic fuzzy transition matrices [9].

In the present study, by taking and worked out the advantageous features of the theories of fuzzy matrices and fuzzy transition matrices the new definition of Product operation on fuzzy transition matrix has been introduced. Further, some properties of trace and transpose are also discussed. In addition, some results are explained with an example in an interesting manner.

Key Words: Fuzzy algebra, fuzzy matrices, fuzzy transition matrices & product operation on fuzzy transition matrix.

1. Preliminaries:

In this section we recall some basic definitions and results which will be needed in the sequel.

Definition 1.1: [2] By a fuzzy matrix we mean a matrix over the fuzzy algebra $\mathcal{F}$.

Definition 1.2: [2] Let $V_{m \times n}$ denote set of all $m \times n$ fuzzy matrices over the fuzzy algebra $\mathcal{F}$.

The operations (+, .) are defined on $V_{m \times n}$ as follows:

(1) For any two elements $A=[a_{ij}]$ and $B=[b_{ij}]\in V_{m\times n}$.

Define $A+B=(sup\{a_{ij},b_{ij}\})=\vee(a_{ij},b_{ij})$ where each $a_{ij},b_{ij}\in F$.

(2) For any element $A=(a_{ij})\in V_{m\times n}$ and a scalar $k\in F$.

Define $kA=(inf\{k,a_{ij}\})=\wedge(k,a_{ij})$.

The system $V_{m\times n}$ together with these operations of component wise fuzzy addition and fuzzy multiplication is called fuzzy vector space over F.

Definition 1.3: [2] Define two operations + and . on the $V_{m \times n}$ as follows

(1) For any two elements $A=[a_{ij}]$ and $B=[b_{ij}]\in V_{m\times n}$.

Define $A+B=(sup\{a_{ij},b_{ij}\})=\vee(a_{ij},b_{ij})$ where each $a_{ij},b_{ij}\in F$.

(2) $AB=(Sup\{inf\{a_{ij},b_{ij}\}\})=\vee\{\wedge(a_{ij},b_{ij})\}$.

Definition 1.4: [2] For $A\in\mathcal{F}_{mn}$ the transpose is obtained by interchanging its rows and columns and is denoted by A^T.

Definition 1.5: [8] Let $V_{2\times 2}$ the set of all 2×2 transition matrix over the fuzzy algebra $\mathcal{F}$. The operation (+, .) are defined on $V_{2\times 2}$ as follows.

For any two elements $A=[a_{ij}]$ and $B=[b_{ij}]\in V_{2\times 2}$.

$$(i)A+B=\begin{cases} a_{ij}+b_{ij}, \text{if} a_{ij}+b_{ij}<1 \\ a_{ij}+b_{ij}-1, \text{if} a_{ij}+b_{ij}>1 \end{cases}$$

$$(ii)A+B=\begin{cases} [1,0]_{1\times 2}, \text{if} a_{ij}+b_{ij}=1 \text{ in row 1} \\ [0,1]_{1\times 2}, \text{if} a_{ij}+b_{ij}=1 \text{ in row 2} \end{cases}$$

(iii) For any scalar $k\in(0,1), kA=\text{frational part of }(10ka_{ij})$.

The system $V_{2\times 2}$ together with these operations of component wise transition addition and multiplication is called as fuzzy transition matrix. It is denoted by $(V_{2\times 2},\mathcal{F})$.

Example 1.6: [8] Let $A=\begin{bmatrix}0.1 & 0.9\\ 0.6 & 0.4\end{bmatrix}$ and $B=\begin{bmatrix}0.7 & 0.3\\ 0.2 & 0.8\end{bmatrix}$ be the fuzzy transition matrices over $\mathcal{F}$. Then $A+B=\begin{bmatrix}0.8 & 0.2\\ 0.8 & 0.2\end{bmatrix}$ is a fuzzy transition matrix.

2. Product Fuzzy Transition Matrices:

We now enter into our new notion of product fuzzy transition matrix as follows.

Definition 2.1: Let $[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$ and $[b_{ij}]=\begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m}\\ b_{21} & b_{22} & \cdots & b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{i1} & b_{i2} & \cdots & b_{im}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nm}\end{bmatrix}$

be two m x n fuzzy transition matrices. Then the product of them is defined by,

$$(i)[a_{ij}].[b_{ij}]=\begin{bmatrix}\sum_{k=1}^{n}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{n}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{n}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right)\\ \sum_{k=1}^{n}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{n}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{n}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \sum_{k=1}^{n}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{n}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{n}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right)\end{bmatrix}$$

(ii) If in the i^{th} row all the corresponding summation values are equal to 1 then,

$$[a_{ij}].[b_{ij}]=\begin{cases}\sum_{k=1}^{n}\left(a_{ik}\,\mathcal{A}_{(\vee,\wedge)}\,b_{ki}\right)=1, if\ i=k\\ \sum_{k=1}^{n}\left(a_{ik}\,\mathcal{A}_{(\vee,\wedge)}\,b_{ki}\right)=0, if\ i\neq k.\\ where\, i=1,2,\ldots,m.\end{cases}$$

where $\mathcal{A}_{(\vee,\wedge)}=\begin{cases}a_{mk}\vee b_{km} & \text{if k is odd.}\\ a_{mk}\wedge b_{km} & \text{if k is even.}\end{cases}$

Example 2.2: Let $A=[a_{ij}]=\begin{bmatrix}0.3 & 0.5 & 0.2\\ 0.6 & 0.1 & 0.3\\ 0.2 & 0.6 & 0.2\\ 0.1 & 0 & 0.9\end{bmatrix}_{4\times 3}$, $B=[b_{ij}]=\begin{bmatrix}0.2 & 0.4 & 0 & 0.4\\ 0.5 & 0.3 & 0.1 & 0.1\\ 0.3 & 0.2 & 0.2 & 0.3\end{bmatrix}_{3\times 4}$.

Then the product of $[a_{ij}].[b_{ij}]=\begin{bmatrix}0.1 & 0.9 & 0.6 & 0.7\\ 0 & 1 & 0 & 0\\ 0 & 0.9 & 0.5 & 0.8\\ 0.1 & 0.3 & 0 & 0.3\end{bmatrix}_{4\times 4}$

Definition 2.3: Let $[a_{ij}]=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ a_{i1} & a_{i2} & \cdots & a_{in}\\ \vdots & \vdots & \ddots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$ and $[b_{ij}]=\begin{bmatrix}b_{11} & b_{12} & \cdots & b_{1m}\\ b_{21} & b_{22} & \cdots & b_{2m}\\ \vdots & \vdots & \ddots & \vdots\\ b_{i1} & b_{i2} & \cdots & b_{im}\\ \vdots & \vdots & \ddots & \vdots\\ b_{n1} & b_{n2} & \cdots & b_{nm}\end{bmatrix}$

be two $m\times m$ fuzzy transition matrices. Then the product of them is defined by,

$$(i)[a_{ij}].[b_{ij}]=\begin{bmatrix} \sum_{k=1}^{m}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{m}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{m}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \\ \sum_{k=1}^{m}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{m}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{m}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{m}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1}^{m}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1}^{m}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \end{bmatrix}.$$

(ii) If in the i^{th} row all the corresponding summation values are equal to 1 then,

$$[a_{ij}].[b_{ij}]=\begin{cases} \sum_{k=1}^{m}\left(a_{ik}\,\mathcal{A}_{(\vee,\wedge)}\,b_{ki}\right)=1, if\ i=k \\ \sum_{k=1}^{m}\left(a_{ik}\,\mathcal{A}_{(\vee,\wedge)}\,b_{ki}\right)=0, if\ i\neq k. \\ where\, i=1,2,\ldots,m. \end{cases} \quad \text{where } \mathcal{A}_{(\vee,\wedge)}=\begin{cases} a_{mk}\vee b_{km} & \text{if k is odd.} \\ a_{mk}\wedge b_{km} & \text{if k is even.} \end{cases}$$

Example 2.4: Let $[a_{ij}]=\begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}, [b_{ij}]=\begin{bmatrix} 0.3 & 0.7 \\ 0.5 & 0.5 \end{bmatrix}$. Then the product of $[a_{ij}].[b_{ij}]=\begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.1 \end{bmatrix}$.

Remark 2.5: The product of two fuzzy transition matrices is not a fuzzy transition matrix.

Example 2.6: Let $[a_{ij}]=\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0.1 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}, [b_{ij}]=\begin{bmatrix} 0.3 & 0.4 & 0.3 \\ 0.1 & 0.2 & 0.7 \\ 0 & 0.9 & 0.1 \end{bmatrix}$.

Then their product $[a_{ij}].[b_{ij}]=\begin{bmatrix} 0.9 & 0.5 & 0 \\ 0 & 0.6 & 0.4 \\ 0.1 & 0.6 & 0.2 \end{bmatrix}$ is not a fuzzy transition matrix.

3. Trace and Transpose of Fuzzy Transition Matrices:

Definition 3.1: Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m\times m$. Then trace of fuzzy transition fuzzy matrix is $tr\mathcal{A}=\sum_{i=1}^{m}a_{ii}=a_{11}+a_{22}+a_{33}+\cdots+a_{mm}$.

Example 3.2: Let $A=[a_{ij}]=\begin{bmatrix} 0.4 & 0.2 & 0.4 \\ 0.6 & 0.1 & 0.3 \\ 0.2 & 0.1 & 0.7 \end{bmatrix}$ be a fuzzy transition matrix of order $m\times m$ then trace of this matrix is $tr\mathcal{A}=0.4+0.1+0.7=0.2$.

Definition 3.3: Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m\times n$. Then the transpose of $[a_{ij}]$ is defined by $[a_{ij}]^{T}=[a_{ji}]_{n\times m}$.

Example 3.4: Let $[a_{ij}]=\begin{bmatrix}0.6 & 0.4\\0.3 & 0.7\\0.8 & 0.2\\0 & 1\end{bmatrix}_{4\times 2}$, then $[a_{ij}]^T=\begin{bmatrix}0.6 & 0.3 & 0.8 & 0.0\\0.4 & 0.7 & 0.2 & 1\end{bmatrix}_{2\times 4}$.

Definition 3.5: Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m\times m$ is said to be a symmetric if $a_{ij}=a_{ji}$ for all i and j.

Proposition 3.6: Let A be the fuzzy transition square matrix of order $m\times m$ and $k\in(0,1)$ be a scalar. Then

$(i) tr(k\mathcal{A})=k(tr\mathcal{A})$.

$(ii) tr\mathcal{A}=tr\mathcal{A}^T$.

Proof:

(i) Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m\times m$. $(i.e) A=[a_{ij}]_{m\times m}$.

$kA=[ka_{ij}]_{m\times m}$

$$tr(k\mathcal{A})=\sum_{i=1}^{m}ka_{ii}.$$

$$=k\sum_{i=1}^{m}a_{ij}.$$

$$=k(tr\mathcal{A}).$$

(ii) Let $A=[a_{ij}]$

$$tr\mathcal{A}=\sum_{i=1}^{m}a_{ii}=a_{11}+a_{22}+a_{33}+\cdots+a_{mm}.$$

$$A^T=[a_{ji}]=B=[b_{ij}].$$

$$tr\mathcal{A}^T=tr\mathcal{B}=\sum_{i=1}^{m}b_{ii}=b_{11}+b_{22}+b_{33}+\cdots+b_{mm}.$$

$$=a_{11}+a_{22}+a_{33}+\cdots+a_{mm}.$$

$$=\sum_{i=1}^{m}a_{ii}. =tr\mathcal{A}.$$

Hence, $tr\mathcal{A}=tr\mathcal{A}^T$.

Proposition 3.7: Let A and B be two fuzzy transition matrices of order $m\times m$, then $tr\mathcal{AB}\neq tr\mathcal{BA}$.

Proof:

Let $A=[a_{ij}], B=[b_{ij}]$ be two fuzzy transition matrices of order $m\times m$.

Then, $A.B=[c_{ij}]$.

$$tr(\mathcal{AB})=\sum_{i=1}^{m}c_{ii}=c_{11}+c_{22}+c_{33}+\cdots+c_{mm}.$$

$$B.A=[d_{ij}]$$

$tr(\mathcal{B}\mathcal{A}) = \sum_{i=1}^{m} d_{ii} = d_{11} + d_{22} + d_{33} + \cdots + d_{mm}.$

Since, $c_{11} + c_{22} + c_{33} + \cdots + c_{mm} \neq d_{11} + d_{22} + d_{33} + \cdots + d_{mm}.$

$\sum_{i=1}^{m} c_{ii} \neq \sum_{i=1}^{m} d_{ii}.$

Hence, $tr(\mathcal{A}\mathcal{B}) \neq tr(\mathcal{B}\mathcal{A}).$

Proposition 3.8: Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m \times n$ and let $k, l \in (0,1)$. Then the following results hold.

$(i)(kA)^T = k(A)^T$.

$(ii)(k+l)A^T = kA^T + lA^T$.

Theorem 3.9: The transposition operation is reflexive, that is $\left(A^T\right)^T = A$

Proof:

Let $A=[a_{ij}]$ and $A^T=[b_{ij}]$.

Then $[b_{ij}]=[a_{ij}]$ for all pairs (i, j).

The transpose of A^T is $[b_{ij}]^T$ and thus $[a_{ij}]$.

Hence $\left(A^T\right)^T = A.$

Proposition 3.10: The sum of the fuzzy transition matrix of order $m \times m$ and its transpose is a symmetric matrix. $(i.e) A + A^T$ is a symmetric matrix.

Proof:

Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m \times m$. Then $A + A^T = [b_{ij}] = B$.

Since $[b_{ij}]=[b_{ji}]=[b_{ij}]^T$. $(i.e) B = B^T$.

Hence B is a symmetric matrix.

Theorem 3.11: The transpose of the sum of the two fuzzy transition matrices of order $m \times n$ is equal to the sum of their transposes, $(i.e)(A+B)^T = A^T + B^T = B^T + A^T$.

Proof:

Let $A=[a_{ij}], B=[b_{ij}]$ be two fuzzy transition matrices of order $m \times n$.

Then $A + B = C, C = [c_{ij}]$ where $[c_{ij}] = [a_{ij}] + [b_{ij}]$ for all pairs (i, j).

Let $A^T = [a_{ji}], B^T = [b_{ji}]$.

Then $\left(A+B\right)^T = [c_{ji}].$

$= \left[a_{ji} + b_{ji}\right]. = A^T + B^T.$

Hence $(A+B)^T = A^T + B^T$.

Remark 3.12: In general $\left(A_1 + A_2 + \ldots + A_n\right)^T = A_1^T + A_2^T + \ldots + A_n^T$.

Theorem 3.13: The transpose of the product of two fuzzy transition matrices is equal to the product of their transpose in reverse order. $(i.e)(A.B)^T = B^T.A^T$.

Proof:

Let $A=[a_{ij}], B=[b_{ij}]$ be two fuzzy transition matrices of order $m \times m$.

Let $AB = [c_{ij}]$.

Then $[c_{ij}] = \sum_{k=1}^{n}\left(a_{ik}\mathcal{A}_{(\vee,\wedge)}b_{kj}\right)$ and is the elements of the j^{th} row and i^{th} column of $[c_{ij}]^T$. $(i.e)(AB)^T$.

The elements $b_{1j}, b_{2j}, \ldots, b_{nj}$ of the j^{th} column of B are the elements of j^{th} row of B^T.

The elements $a_{i1}, a_{i2}, \ldots, a_{in}$ of the i^{th} row of A are the elements of i^{th} column of A^T.

The elements of j^{th} row and i^{th} column of $B^T.A^T$ is $\sum_{k=1}^{n}\left(b_{kj}\mathcal{A}_{(\vee,\wedge)}a_{ij}\right)$.

That is $\sum_{k=1}^{n}\left(a_{ik}\mathcal{A}_{(\vee,\wedge)}b_{kj}\right)$.

Hence, $(A.B)^T = B^T.A^T$.

Theorem 3.14: The sum of any fuzzy transition matrix and its transpose is a symmetric matrix; that is $(A+A^T)^T = (A+A^T)$.

Proof:

$(A+A^T)^T = A^T + (A^T)^T$; by Theorem 3.11.

$= A^T + A$; by Theorem 3.9.

$= A + A^T$ since the addition of matrices is commutative.

Hence $A+A^T$ is a symmetric matrix.

Theorem 3.15: Let $A=[a_{ij}]$ be a fuzzy transition matrix of order $m\times 2$, then $(A^TA)^T = A^TA$, is a symmetric matrix.

Proof:

$(A^TA)^T = A^T(A^T)^T\left[since(AB)^T = B^T.A^T\right]$.

$= A^TA\left[since(A^T)^T = A\right]$.

Hence, A^TA is a symmetric matrix.

Remark 3.16: If A is a fuzzy transition matrix of order $m\times 2$, then AA^T is a identity matrix of order m; and A^TA is a symmetric matrix of order 2. It is shown by the following theorems.

Theorem 3.17: The product of $A.A^T$ of two fuzzy transition matrices always an identity matrix. $(i.e)[a_{ij}]_{m\times 2}.[a_{ji}]_{2\times m} = I_m =$ identity matrix.

Proof:

Let $A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix}_{m\times 2}$, then the transpose of $[a_{ij}]$ is

$$A^T = [a_{ij}]^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \end{bmatrix}_{2\times m}.$$

Then the product of $A.A^T$ = $[a_{ij}].[a_{ij}]^T$ is,

$$[a_{ij}].[a_{ij}]^T = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix}_{m\times 2} . \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \end{bmatrix}_{2\times m},$$

$$= \begin{bmatrix} \sum_{k=1,2}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1,2}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1,2}\left(a_{1k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \\ \sum_{k=1,2}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1,2}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1,2}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1,2}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k1}\right) & \sum_{k=1,2}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{k2}\right) & \cdots & \sum_{k=1,2}\left(a_{mk}\,\mathcal{A}_{(\vee,\wedge)}\,b_{km}\right) \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}_{m\times m} = I_m \text{(a Identity matrix)[since by the definition 2.3 (ii)].}$$

Theorem 3.18: The product of $A^T A$ of two fuzzy transition matrix is always a symmetric matrix.

Proof:

Let $A=[a_{ij}]=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix}_{m\times 2}$, then the transpose of $[a_{ij}]$ is

$$A^T = [a_{ij}]^T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \end{bmatrix}_{2\times m}.$$

Then the product of $A^T A = [a_{ij}]^T.[a_{ij}]$ is,

$$[a_{ij}]^T.[a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{m1} \\ a_{21} & a_{22} & \cdots & a_{m2} \end{bmatrix}_{2\times m} . \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \vdots & \vdots \\ a_{m1} & a_{m2} \end{bmatrix}_{m\times 2}.$$

$$= \begin{bmatrix} \sum_{k=1}^{m}\left(a_{k1}\,\mathcal{A}_{(\vee,\wedge)}\,a_{k1}\right) & \sum_{k=1}^{m}\left(a_{k1}\,\mathcal{A}_{(\vee,\wedge)}\,a_{2k}\right) \\ \sum_{k=1}^{m}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,a_{k1}\right) & \sum_{k=1}^{m}\left(a_{2k}\,\mathcal{A}_{(\vee,\wedge)}\,a_{2k}\right) \end{bmatrix}.$$

$$=[b_{ij}]_{2\times 2} = B\left(is\,a\,2\times 2 \text{ symmetric matrix}\right). \quad [\text{Since } B=B^T].$$

Theorem 3.19: If A is a fuzzy transition matrix of order $m\times m$, then pre and post product of A with A^T is a fuzzy transition matrix of order $m\times m$. $(i.e)AA^T A$ is a fuzzy transition matrix of order $m \times m$.

Proof:

Let $A = [a_{ij}]_{m\times m}$, then the transpose of $[a_{ij}]$ is $[a_{ji}] = A^T$.

$AA^T = [a_{ij}].[a_{ij}]^T$.

$AA^T = I_m$, by the Theorem 3.17

$AA^T A = (AA^T)A.$

$= I_m.A = I_m.[a_{ij}].$

$= [b_{ij}] = B.$

Hence B is a fuzzy transitive matrix of order $m \times m$.

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