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SOME CHARACTERIZATIONS ON CUBIC SETS

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Abstract:

*In this paper we have defined * cubic sets and discussed some of their characterizations.*

Index Terms: Cubic Set, Internal Cubic Set, External Cubic Sets, P-(R-) Order, P-(R-) Union & P-(R-) Intersection.

Introduction:

In 1965, Zadeh [6] has initiated the notion of fuzzy sets and Zadeh [6] made an extension of the concept of a fuzzy set by an interval-valued fuzzy set. In traditional fuzzy logic, the expert's degrees of certainty in different statements were given by numbers from the interval [0, 1]. It is often difficult for an expert to exactly quantify his or her certainty. Therefore, instead of a real number, it is more adequate to represent this degree of certainity by an interval or even by a fuzzy set. Some properties of Fuzzy Ideals in Algebraic structures were discussed by Chinnadurai [1]. Interval-valued fuzzy sets have been actively used in real-life applications. Sambuc [3] used this in Medical diagnosis in thyroidian pathology. Kohout [2] also used this in Medicine, in a system CLINAID, Gorzalczany in Approximate reasoning. Turksen [4] in Interval-valued logic, in preferences modelling. These works show the importance of cubic sets. Cubic sets were discussed by Jun [5]

In this paper, using a some properties of Cubic ideals of near rings studied by Chinnadurai fuzzy set and an interval-valued fuzzy set, we introduce a new notion, called (interval, external) * cubic set, and investigate several properties. We particularly it deals with P-union, P-intersection, R-union, R-intersection of * cubic sets, and related properties.

Definition: 1.1 Let X be a nonempty set. A fuzzy set $\mathcal{A}$ in X is characteristic by its membership function (generalized characteristic function).

$$\mu_A : X \rightarrow [0, 1]$$

And $\mu_A(x)$ is interpreted as its degree of membership elements x in a fuzzy set A for each x$\in X$. i.e. [0, 1] is the subset of nonnegative real numbers whose supremum is finite.

Definition: 1.2 Let X be any set. A mapping $\overline{A} : X \rightarrow D[0,1]$ is called an interval- valued fuzzy subset (briefly, i-v fuzzy subset) of X, where D[0,1] denotes the family of all closed subintervals of [0, 1] and $\overline{A}$ (x)= [A^-(x), A^+(x)] for all x$\in X$, where A^- and A^+ are fuzzy subsets of X such that $A^-(x) \leq A^+(x)$ for all x$\in X$.

Definition: 1.3 Let X be a nonempty set. A cubic set $\mathcal{A}$ and $\mathcal{B}$ is a structure of the form

$$\mathcal{A} = \{< x, A(x), \lambda(x)> | \ x \in X\} \text{ and } \mathcal{B} = \{< x, B(x), \mu(x)> | \ x \in X\}$$

Then by define * property

$$\mathcal{A}^* = \{< x, A(x), \mu(x)> | \ x \in X\} \text{ and } \mathcal{B}^* = \{< x, B(x), \lambda(x)> | \ x \in X\}$$

in which A and B is an interval valued fuzzy set (IVF) in X and λ and μ is a fuzzy set in X.

A cubic set $\mathcal{A}^* = \{< x, A(x), \mu(x)> | \ x \in X\}$ is simply denoted by $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^* = \{< x, B(x), \lambda(x)> | \ x \in X\}$ is simply denoted by $\mathcal{B}^*$= <B, λ>.

Definition: 1.4 Let X be a nonempty set. A cubic set $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^*$ = <B, λ> in X said to be internal cubic set (briefly, ICS) if $A^-(x) \leq \mu(x) \leq A^+(x)$ and $B^-(x) \leq \lambda(x) \leq B^+(x)$ for all $x \in X$.

Definition: 1.5 Let X be a nonempty set. A cubic set $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^*$ = <B, λ> in X said to be external cubic set (briefly, ECS) if $\mu(x) \notin (A^-(x), A^+(x))$ and $\lambda(x) \notin (B^-(x), B^+(x))$ for all $x \in X$.

Example: 1.6 Let X = {a, b} be a nonempty set. A cubic set $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A}$ = {< a, [0.1, 0.3], 0.5 >, < b, [0.2, 0.4], 0.6>} and $\mathcal{B}$ = {< a, [0.5,0.7], 0.6 >, < b, [0.8, 0.9], 0.3 >} then * property as $\mathcal{A}^*$ = {< a, [0.1, 0.3], 0.6>, < b, [0.2, 0.4], 0.3>} and $\mathcal{B}^*$ = {< a, [0.5,0.7], 0.5 >, < b, [0.8, 0.9], 0.6 >}.

Example: 1.7 Let $\mathcal{A}$ = {< x, A(x), λ(x)>| x ∈ X } and $\mathcal{B}$ = {< x, B(x), μ(x)>| x ∈ X } be a cubic set in X. If A(x) = [0.2, 0.7], $\lambda(x)$ =0.5 and B(x) = [0.4, 0.6], μ(x) = 0.6 for all $x \in X$, then $\mathcal{A}^*$ and $\mathcal{B}^*$ is an ICS.

Example: 1.8 Let $\mathcal{A}$ = {< x, A(x), λ(x)>| x ∈ X } and $\mathcal{B}$ = {< x, B(x), μ(x)>| x ∈ X } be a cubic set in X. If A(x) = [0.2, 0.5], $\lambda(x)$ =0.9 and B(x) = [0.6, 0.8], μ(x) = 0.1 for all $x \in X$, then $\mathcal{A}^*$ and $\mathcal{B}^*$ is an ICS.

Theorem: 1.9 Let $\mathcal{A}$ = {< x, A(x), λ(x)>| x ∈ X } and $\mathcal{B}$ = {< x, B(x), μ(x)>| x ∈ X } be a cubic sets in X. If $\mathcal{A}$ and $\mathcal{B}$ is both an ICS and an ECS. Also let $\mathcal{A}^*$ and $\mathcal{B}^*$ is both an ICS and an ECS, then $\mu(x) \in U(A) \cup L(A)$ and $\lambda(x) \in U(A) \cup L(A)$ (for all x ∈ X) where U(A) = { $A^+(x)$ | x ∈ X} and L(A) = { $A^-(x)$ | x ∈ X }.

Proof: Assume that $\mathcal{A}$ and $\mathcal{B}$ is both an ICS and ECS. Also $\mathcal{A}^*$ and $\mathcal{B}^*$ is both an ICS and ECS. Using definition of ICS and ECS and we have $A^-(x) \leq \mu(x) \leq A^+(x)$, $B^-(x) \leq \lambda(x) \leq B^+(x)$ and $\mu(x) \notin (A^-(x), A^+(x))$ and $\lambda(x) \notin (B^-(x), B^+(x))$ for all $x \in X$. Thus $\mu(x) = A^-(x)$ or, $A^+(x) = \mu(x), \lambda(x) = B^-(x)$ or $B^+(x) = \lambda(x)$ and so $\mu(x) \in U(A) \cup L(A), \lambda(x) \in U(A) \cup L(A)$.

Definition: 1.10 Let $\mathcal{A}$ = < A, λ> and $\mathcal{B}$ = < B, μ> be cubic sets in X. Also $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^*$ = <B, λ> be a cubic sets in X. Then we define

(a) (Equality) $\mathcal{A}^* = \mathcal{B}^* \Leftrightarrow$ A = B and μ = λ.
(b) (P-order) $\mathcal{A}^* \subseteq_P \mathcal{B}^* \Leftrightarrow A \subseteq B$ and μ ≤ λ.
(c) (R-order) $\mathcal{A}^* \subseteq_R \mathcal{B}^* \Leftrightarrow A \subseteq B$ and μ ≥ λ.

Definition: 1.11 For any $\mathcal{A}_i$= {<x, $\mathcal{A}_i$(x), λ_i(x)>x∈ X} and $\mathcal{B}_i$=**{**<x, $\mathcal{B}_i$(x), μ_i(x)>|x∈X}. Also $\mathcal{A}_i^*$={<x, A_i(x), μ_i(x)>| x∈X} and $\mathcal{B}_i^*$={<x, B_i(x), λ_i(x)>| x∈X} be a cubic sets where i∈Ω, and i taken as even sets only we define

(a) P-union

$$\bigcup_{(i\in\Omega)} {}_P\mathcal{A}_i^* = \{<x, (\bigcup_{(i\in\Omega)} A_i)(x), (\bigvee_{(i\in\Omega)} \mu_i)(x)> |x\in X\}$$

$$\bigcup_{(i\in\Omega)} {}_P\mathcal{B}_i^* = \{<x, (\bigcup_{(i\in\Omega)} B_i)(x), (\bigvee_{(i\in\Omega)} \lambda_i)(x)> |x\in X\}$$

(b) P-intersection

$$\bigcap_{(i\in\Omega)} {}_P\mathcal{A}_i^* = \{<x, (\bigcap_{(i\in\Omega)} A_i)(x), (\bigwedge_{(i\in\Omega)} \mu_i)(x)> |x\in X\}$$

$$\bigcap_{(i\in\Omega)} {}_P\mathcal{B}_i^* = \{<x, (\bigcap_{(i\in\Omega)} B_i)(x), (\bigwedge_{(i\in\Omega)} \lambda_i)(x)> |x\in X\}$$

(c) R-union

$$\bigcup_{(i\in\Omega)} {}_R\mathcal{A}_i^* = \{<x, (\bigcup_{(i\in\Omega)} A_i)(x), (\bigvee_{(i\in\Omega)} \mu_i)(x)> |x\in X\}$$

$\bigcup_{(i\in\Omega)} {}_R\mathcal{B}_i^* = \{<x, (\bigcup_{(i\in\Omega)} B_i)(x), (\vee_{(i\in\Omega)} \lambda_i)(x)> | x\in X\}$

(d) P-intersection

$\bigcap_{(i\in\Omega)} {}_R\mathcal{A}_i^* = \{<x, (\bigcap_{(i\in\Omega)} A_i)(x), (\wedge_{(i\in\Omega)} \mu_i)(x)> | x\in X\}$

$\bigcap_{(i\in\Omega)} {}_R\mathcal{B}_i^* = \{<x, (\bigcap_{(i\in\Omega)} B_i)(x), (\wedge_{(i\in\Omega)} \lambda_i)(x)> | x\in X\}$

The complement of $\mathcal{A}$ = < A, λ> and $\mathcal{B}$ = < B, μ> be cubic sets in X. Also $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^*$ = <B, λ> is defined to be a cubic sets $\mathcal{A}^{*c}$={< x, $A^c(x)$, 1- μ(x)>| x ∈ X } and $\mathcal{B}^{*c}$={< x, $B^c(x)$, 1- λ(x)>| x ∈ X } where $A^c(x) = [1- A^+(x), 1- A^-(x)]$ and $B^c(x) = [1- B^+(x), 1- B^-(x)]$.

Theorem: 1.12 Let $\mathcal{A}$ = < A, λ> and $\mathcal{B}$ = < B, μ> be cubic sets in X. Also $\mathcal{A}^*$ = <A, μ> and $\mathcal{B}^*$ = <B, λ> be a cubic sets. Then the following is true $(\mathcal{A}^{*c})^c = \mathcal{A}^*$ and $(\mathcal{B}^{*c})^c = \mathcal{B}^*$.

Proof: Let $\mathcal{A}^{*c}$={< x, $A^c(x)$, 1- μ(x)>| x ∈ X }

$(\mathcal{A}^{*c})^c$={< x, $(A^c(x))^c$, $(1- \mu(x))^c$>| x ∈ X }

={< x, A(x),1- 1- μ(x)>| x ∈ X }

=$\mathcal{A}$

Similllary we can prove that $(\mathcal{B}^{*c})^c = \mathcal{B}^*$.

Theorem: 1.13 For any $\mathcal{A}_i$= {<x, $\mathcal{A}_i(x)$, $\lambda_i(x)$>x∈ X} and $\mathcal{B}_i$**=**{<x, $\mathcal{B}_i(x)$, $\mu_i(x)$>|x∈X}. Also $\mathcal{A}_i^*$={<x, $A_i(x)$, $\mu_i(x)$>| x∈X} and $\mathcal{B}_i^*$={<x, $B_i(x)$, $\lambda_i(x)$>| x∈X} , where i∈Ω family of cubic sets.

$(\mathcal{A}_1^* \cup_P \mathcal{A}_2^* \ldots \cup_P \mathcal{A}_n^*)^c = \mathcal{A}_1^{*c} \cup_P \mathcal{A}_2^{*c} \cup_P \ldots \cup_P \mathcal{A}_n^{*c}$,
$\mathcal{A}_1^{*c} \cap_P \mathcal{A}_2^{*c} \cap_P \ldots \cap_P \mathcal{A}_n^{*c} = (\mathcal{A}_1^* \cap_P \mathcal{A}_2^* \ldots \cap_P \mathcal{A}_n^*)^c$

We have $\left(\bigcup_{(i\in\Omega)} P\mathcal{A}_i^*\right)^c = \left(\bigcap_{(i\in\Omega)} P\mathcal{A}_i^{*c}\right)$

$(\mathcal{B}_1^* \cup_P \mathcal{B}_2^* \ldots \cup_P \mathcal{B}_n^*)^c = \mathcal{B}_1^{*c} \cup_P \mathcal{B}_2^{*c} \cup_P \ldots \cup_P \mathcal{B}_n^{*c}$,
$\mathcal{B}_1^{*c} \cap_P \mathcal{B}_2^{*c} \cap_P \ldots \cap_P \mathcal{B}_n^{*c} = (\mathcal{B}_1^* \cap_P \mathcal{B}_2^* \ldots \cap_P \mathcal{B}_n^*)^c$

We have $\left(\bigcup_{(i\in\Omega)} P\mathcal{B}_i^*\right)^c = \left(\bigcap_{(i\in\Omega)} P\mathcal{B}_i^{*c}\right)$.

$(\mathcal{A}_1^* \cap_P \mathcal{A}_2^* \ldots \cap_P \mathcal{A}_n^*)^c = \mathcal{A}_1^{*c} \cap_P \mathcal{A}_2^{*c} \cap_P \ldots \cap_P \mathcal{A}_n^{*c}$,
$\mathcal{A}_1^{*c} \cup_P \mathcal{A}_2^{*c} \cup_P \ldots \cup_P \mathcal{A}_n^{*c} = (\mathcal{A}_1^* \cup_P \mathcal{A}_2^* \ldots \cup_P \mathcal{A}_n^*)^c$

We have $\left(\bigcap_{(i\in\Omega)} P\mathcal{A}_i^*\right)^c = \left(\bigcup_{(i\in\Omega)} P\mathcal{A}_i^{*c}\right)$

$(\mathcal{B}_1^* \cup_P \mathcal{B}_2^* \ldots \cup_P \mathcal{B}_n^*)^c = \mathcal{B}_1^{*c} \cup_P \mathcal{B}_2^{*c} \cup_P \ldots \cup_P \mathcal{B}_n^{*c}$,
$\mathcal{B}_1^{*c} \cap_P \mathcal{B}_2^{*c} \cap_P \ldots \cap_P \mathcal{B}_n^{*c} = (\mathcal{B}_1^* \cap_P \mathcal{B}_2^* \ldots \cap_P \mathcal{B}_n^*)^c$

We have $\left(\bigcap_{(i\in\Omega)} P\mathcal{B}_i^*\right)^c = \left(\bigcup_{(i\in\Omega)} P\mathcal{B}_i^{*c}\right)$.

Also similarly we have $\left(\bigcup_{(i\in\Omega)} R\mathcal{A}_i^*\right)^c = \left(\bigcap_{(i\in\Omega)} R\mathcal{A}_i^{*c}\right)$, $\left(\bigcup_{(i\in\Omega)} R\mathcal{B}_i^*\right)^c = \left(\bigcap_{(i\in\Omega)} R\mathcal{B}_i^{*c}\right)$. $\left(\bigcap_{(i\in\Omega)} R\mathcal{A}_i^*\right)^c = \left(\bigcup_{(i\in\Omega)} R\mathcal{A}_i^{*c}\right)$, have $\left(\bigcap_{(i\in\Omega)} R\mathcal{B}_i^*\right)^c = \left(\bigcup_{(i\in\Omega)} R\mathcal{B}_i^{*c}\right)$.

Proof: Straightforward.

Proposition: 1.14 Let $\mathcal{A}$ = < A, λ>, $\mathcal{B}$ = < B, μ>, $\mathcal{C}$ = <C, γ>, $\mathcal{D}$= <D,ρ>be cubic sets. Also $\mathcal{A}^*$ = < A, μ>, $\mathcal{B}^*$ = < B, λ>, $\mathcal{C}^*$= <C, ρ>, $\mathcal{D}^*$= <D,γ>be cubic sets. Then the following statements are true.

(1) If $\mathcal{A}^* \subseteq_P \mathcal{B}^*$ and $\mathcal{B}^* \subseteq_P \mathcal{C}^*$ then $\mathcal{A}^* \subseteq_P \mathcal{B}^*$.
(2) If $\mathcal{A}^* \subseteq_P \mathcal{B}^*$ and $\mathcal{B}^*$
(3) If $\mathcal{A}^* \subseteq_P \mathcal{B}^*$ and $\mathcal{A}^* \subseteq_P \mathcal{C}^*$ then $\mathcal{A}^* \subseteq_P \mathcal{B}^* \cap_R \mathcal{C}^*$.
(4) If $\mathcal{A}^* \subseteq_P \mathcal{B}^*$ and $\mathcal{C}^* \subseteq_P \mathcal{B}^*$ then $\mathcal{A}^* \cup_P \mathcal{C}^* \subseteq_P$ B*.
(5) If $\mathcal{A} * \subseteq_P \mathcal{B}^*$ and $\mathcal{C}^* \subseteq_P \mathcal{D}^*$ then $\mathcal{A}^* \cup_P \mathcal{C}^* \subseteq_P \mathcal{C}^* \cup_P \mathcal{D}^*$ and $\mathcal{A}^* \cap_P \mathcal{C}^* \subseteq_P \mathcal{B}^* \cap_P \mathcal{D}^*$.

Proof: Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be cubic sets. Also $\mathcal{A}^*, \mathcal{B}^*$, $\mathcal{C}^*$, $\mathcal{D}^*$ be cubic sets. By definition,

(1) $\mathcal{A} * \subseteq_P \mathcal{B}^* \Rightarrow$ A $\subseteq$ B and μ ≤ λ.
$\mathcal{B}^* \subseteq_P \mathcal{C}^* \Rightarrow$ B $\subseteq$ C and λ ≤ ρ.
$\Rightarrow$ A $\subseteq$ B $\subseteq$ C and λ ≤ μ ≤ γ
$\Rightarrow$ A $\subseteq$ C and λ ≤ γ.
$\Rightarrow \mathcal{A}^* \subseteq_P \mathcal{C}^*$.

(2) $A^c \supseteq B^c$ and μ ≤ λ.
$B^c \subseteq A^c$ and λ ≥ μ.
$\mathcal{B}^{*c} \subseteq_P \mathcal{A}^{*c}$

(3) $\mathcal{A} * \subseteq_P \mathcal{B}^* \Rightarrow$ A $\subseteq$ C and μ ≤ ρ.
$C \subseteq B \cup C$ and $\mu \subseteq \lambda \wedge \rho$
A$\subseteq_P$ B$\cup_P$C and $\mu \subseteq \lambda \wedge \rho$.
$\mathcal{A}^* \subseteq_P \mathcal{B}^* \cap_R \mathcal{C}^*$

(4) $\mathcal{A}^* \subseteq_P \mathcal{B}^* \Rightarrow$ A $\subseteq$ B and μ ≤ λ.
$\mathcal{C}^* \subseteq_P \mathcal{B}^* \Rightarrow$ C $\subseteq$ B and ρ ≤ λ .
A ∪ C $\subseteq$ B and μ ∨ ρ ≤ λ.
$\mathcal{A}^* \cup_P \mathcal{C}^* \subseteq_P$ B*.

(5) $\mathcal{A}^* \subseteq_P \mathcal{B}^* \Rightarrow$ A $\subseteq$ B and μ ≤ λ.
and $\mathcal{C}^* \subseteq_P \mathcal{D}^* \Rightarrow$ C $\subseteq$ D and ρ ≤ γ.
$\Rightarrow$ A∪ C $\subseteq$ B∪ D and μ ∨ ρ ≤ λ ∨ γ.
$\mathcal{A}^* \cup_P \mathcal{C}^* \subseteq_P \mathcal{C}^* \cup_P \mathcal{D}^*$
$\Rightarrow$ A ∩ C $\subseteq$ B ∩ D and μ ∧ ρ ≤ λ ∧ γ.
$\Rightarrow \mathcal{A}^* \cap_P \mathcal{C}^* \subseteq_P \mathcal{B}^* \cap_P \mathcal{D}^*$.

Hence proved.

Proposition: 1.15 Let $\mathcal{A}$ = < A, λ>, $\mathcal{B}$ = < B, μ>, $\mathcal{C}$= <C, γ>, $\mathcal{D}$= <D,ρ>be cubic sets. Also $\mathcal{A}^*$=<A, μ>, $\mathcal{B}^*$= < B, λ>, $\mathcal{C}^*$=<C, ρ>, $\mathcal{D}^*$=<D,γ>be cubic sets. Then the following statements are true.

(1) If $\mathcal{A}^* \subseteq_R \mathcal{B}^*$ and $\mathcal{B}^* \subseteq_R \mathcal{C}^*$ then $\mathcal{A}^* \subseteq_R \mathcal{C}^*$.
(2) If $\mathcal{A}^* \subseteq_R \mathcal{B}^*$ and $\mathcal{B}^{*c} \subseteq_R \mathcal{A}^{*c}$
(3) If $\mathcal{A}^* \subseteq_R \mathcal{B}^*$ and $\mathcal{A} * \subseteq_R \mathcal{C}^*$ then $\mathcal{A}^* \subseteq_R \mathcal{B}^* \cap_R \mathcal{C}^*$
(4) If $\mathcal{A}^* \subseteq_R \mathcal{B}^*$ and $\mathcal{C}^* \subseteq_R \mathcal{B}^*$ then $\mathcal{A}^* \cup_R \mathcal{C}^* \subseteq_P$ B*.
(5) If $\mathcal{A}^* \subseteq_R \mathcal{B}^*$ and $\mathcal{C}^* \subseteq_P \mathcal{D}^*$ then $\mathcal{A}^* \cup_R \mathcal{C}^* \subseteq_R \mathcal{C}^* \cup_R \mathcal{D}^*$ and $\mathcal{A}^* \cap_R \mathcal{C}^* \subseteq_R \mathcal{B}^* \cap_R \mathcal{D}^*$.

Proof: Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be cubic sets. Also $\mathcal{A}^*, \mathcal{B}^*$, $\mathcal{C}^*$, $\mathcal{D}^*$ be cubic sets. By definition,

(1) $\mathcal{A}^* \subseteq_R \mathcal{B}^* \Rightarrow$ A $\subseteq$ B and μ ≤ λ.
$\mathcal{B}^* \subseteq_R \mathcal{C}^* \Rightarrow$ B $\subseteq$ C and λ ≤ ρ.
$\Rightarrow$ A $\subseteq$ B $\subseteq$ C and μ ≤ λ ≤ ρ
$\Rightarrow$ A $\subseteq$ C and μ ≤ ρ.

$\Rightarrow \mathcal{A}^* \subseteq_R \mathcal{C}^*$.

(2) $A^c \supseteq B^c$ and μ ≤ λ.

$\Rightarrow B^c \subseteq A^c$ and λ ≤ μ.

$\Rightarrow \mathcal{B}^{*c} \subseteq_R \mathcal{A}^{*c}$

(3) $\mathcal{A}^* \subseteq_R \mathcal{B}^* \Rightarrow$ A ⊆ B and μ ≤ λ.

$\mathcal{A}^* \subseteq_R \mathcal{C}^*$. ⇒ A ⊆ C and μ ≤ ρ.

$C \subseteq B \cup C$ and μ ≤ λ ∧ ρ.

A ⊆ B ∪ C and μ ≤ λ ∧ ρ.

$\Rightarrow \mathcal{A}^* \subseteq_R \mathcal{B}^* \cap_R \mathcal{C}^*$

(4) $\mathcal{A}^* \subseteq_R \mathcal{B}^* \Rightarrow$ A ⊆ B and μ ≤ λ.

$\mathcal{C}^* \subseteq_R \mathcal{B}^* \Rightarrow$ C ⊆ B and ρ ≤ λ .

A ∪ C ⊆ B and μ ∨ ρ ≤ λ.

$\Rightarrow \mathcal{A}^* \cup_R \mathcal{C}^* \subseteq_P$ B*.

(5) $\mathcal{A}^* \subseteq_R \mathcal{B}^* \Rightarrow$ A ⊆ B and μ ≤ λ.

and $\mathcal{C}^* \subseteq_P \mathcal{D}^* \Rightarrow$ C ⊆ D and ρ ≤ γ.

⇒ A ∪ C ⊆ B ∪ D and μ ∨ ρ ≤ λ ∨ γ.

$\mathcal{A}^* \cup_R \mathcal{C}^* \subseteq_R \mathcal{C}^* \cup_R \mathcal{D}^*$

⇒ A ∩ C ⊆ B ∩ D and μ ∧ ρ ≤ λ ∧ γ.

$\Rightarrow \mathcal{A}^* \cap_R \mathcal{C}^* \subseteq_R \mathcal{B}^* \cap_R \mathcal{D}^*$.

Hence proved.

Theorem: 1.16 Let $\mathcal{A}$ = < A, λ> and B = < B, μ> be a cubsic set in X. Also let $\mathcal{A}^*$ = < A, μ>, $\mathcal{B}^*$ = < B, λ> be a cubic set in X . If $\mathcal{A}^*$ is an ICS, then $\mathcal{A}^{*c}$ is an ICS.

Proof: Since $\mathcal{A}^*$ = < A, μ> is an ICS in X, we have $A^-(x) \leq$ μ(x) ≤ $A^+(x)$ for all x ∈ X. This implies that 1-A^+(x) ≤ 1-μ(x) ≤ 1- A^-(x). Hence $\mathcal{A}^{*c}$ = {< x, A^c(x), 1- μ(x) >| x ∈ X } is an ICS in X.

Theorem: 1.17 Let $\mathcal{A}$ = < A, λ> and $\mathcal{A}$ = < B, μ> be a cubic set in X. Also let $\mathcal{A}^*$ = < A, μ>, $\mathcal{B}^*$ = < B, λ> be a cubic set in X . If $\mathcal{A}^*$ is an ECS, then $\mathcal{A}^{*c}$ is an ECS.

Proof: Since $\mathcal{A}^*$ = < A, μ> is an ECS in X, we have $\mu(x) \notin (A^-(x), A^+(x))$ for all x ∈ X. This implies that 1-μ(x) ∉ (1-A^+(x) , 1- A^-(x)). Hence $\mathcal{A}^{*c}$ = {< x, A^c(x), 1- μ(x) >| x ∈ X } is an ECS in X.

Theorem: 1.18 Let $\mathcal{A}_i$ ={ <A_i, λ_i>| i∈Ω} and $\mathcal{B}_i$={<B_i, μ_i>|i∈Ω} be a family of ICSs in X. Also let $\mathcal{A}_i^*$={<A_i, μ_i>| i∈Ω} and $\mathcal{B}_i^*$={<B_i, λ_i>| i∈Ω} be a family of ICSs in X. Then the P-union and P-intersection of $\mathcal{A}$ * = < A, μ>, $\mathcal{B}$ * = < B, λ> are ICSs in X.

Proof: Since $\mathcal{A}_i$ and $\mathcal{B}_i$ is an ICS in X. Also $\mathcal{A}_i^*$ and $\mathcal{B}_i^*$ is an ICS in X , we have A_i^- (x)≤μ_i (x)≤ A_i^+(x) and B_i^- (x)≤λ_i(x)≤ B_i^+(x) for i∈Ω. This implies that

$$\left(\bigcup_{(i\in\Omega)} A_i\right)^-(x) \leq \left(\bigvee_{(i\in\Omega)} \mu_i\right)(x) \leq \left(\bigcup_{(i\in\Omega)} A_i\right)^+(x),$$

$$\left(\bigcup_{(i\in\Omega)} B_i\right)^-(x) \leq \left(\bigvee_{(i\in\Omega)} \lambda_i\right)(x) \leq \left(\bigcup_{(i\in\Omega)} B_i\right)^+(x),$$

and

$$\left(\bigcap_{(i\in\Omega)} A_i\right)^-(x) \leq \left(\bigwedge_{(i\in\Omega)} \mu_i\right)(x) \leq \left(\bigcap_{(i\in\Omega)} A_i\right)^+(x),$$

$$\left(\bigcap_{(i\in\Omega)} B_i\right)^-(x) \leq \left(\bigwedge_{(i\in\Omega)} \lambda_i\right)(x) \leq \left(\bigcap_{(i\in\Omega)} B_i\right)^+(x),$$

Hence then the P-union and P-intersection of $\mathcal{A}^*$ and $\mathcal{B}^*$ are ICSs in X.

Remark: 1.19 The above theorem is not true for ECS in X with respect to P-union and P-intersection. For instances, let $\mathcal{A}^* = < A, \mu>$ and $\mathcal{B}^* = < B, \lambda>$ be ECSs in I = [0,1] in which A(x) = [0.3, 0.5], λ(x) = 0.8, B(x)=[0.7,0.9] and μ(x)= 0.4 for all x ∈ X.

(1) We know that $\mathcal{A}^* \cup_R \mathcal{B}^* = \{< x, B(x), \lambda(x)>| x \in X\}$ and $\lambda(x) \in [B^-(x), B^+(x)]$ for all x ∈ X. Hence $\mathcal{A}^* \cup_R \mathcal{B}^*$ is not an ECS in X.

(2) We know that $\mathcal{A}^* \cap_R \mathcal{B}^* = \{< x, A(x), \mu(x)>| x \in X\}$ and $\mu(x) \in [A^-(x), A^+(x)]$ for all x ∈ X. Hence $\mathcal{A}^* \cap_R \mathcal{B}^*$ is not an ECS in X.

The following examples show that R-union and R-intersection of cubic sets need not be ECSs.

Example: 1.20 Let $\mathcal{A} = < A, \lambda>$ and $\mathcal{B} = < B, \mu>$ be two ECSs. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$ be two ECSs in I = [0,1] in which A(x) = [0.3, 0.5], λ(x) = 0.4, B(x)=[0.7,0.9] and μ(x)= 0.8 for all x ∈ X.

(1) We know that $\mathcal{A}^* \cup_R \mathcal{B}^* = \{< x, B(x), \lambda(x)>| x \in X\}$ and $\lambda(x) \notin [B^-(x), B^+(x)]$ for all x ∈ X. Hence $\mathcal{A}^* \cup_R \mathcal{B}^*$ is not an ICS in X.

(2) We know that $\mathcal{A}^* \cap_R \mathcal{B}^* = \{< x, A(x), \mu(x)>| x \in X\}$ and $\mu(x) \notin [A^-(x), A^+(x)]$ for all x ∈ X. Hence $\mathcal{A}^* \cap_R \mathcal{B}^*$ is not an ICS in X.

The following theorem provides a condition for the R-union of two ICSs to be an ICS.

Theorem: 1.21 Let $\mathcal{A} = < A, \lambda>$ and $\mathcal{B} = < B, \mu>$ be two ICSs in X. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$ be two ICSs in X such that

$$\max\{ A^-(x), B^-(x) \} \leq (\mu \wedge \lambda)(x)$$

for all x ∈ X. Then the R-union of $\mathcal{A}^*$ and $\mathcal{B}^*$ is an ICS in X.

Proof: Let $\mathcal{A} = < A, \lambda>$ and $\mathcal{B} = < B, \mu>$ be ICSs in X. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$, be ICSs in X which satisfy the condition.

Then $A^-(x) \leq ?(x) \leq A^+(x)$ and , $B^-(x) \leq \lambda(x) \leq B^+(x)$ which implies that $(\mu \wedge \lambda)(x) \leq (A \cup B)^+(x)$. It follows from the definition of condition that

$$(A \cup B)^-(x) = \max\{A^-(x), B^-(x)\} \leq (\mu \wedge \lambda)(x) \leq (A \cup B)^+(x)$$

So that $\mathcal{A}^* \cup_R \mathcal{B}^* = \{<x, (A \cup B)(x), (\mu \wedge \lambda)(x) >| x \in X\}$ is an ICS in X.

The following theorem provide a condition for the R-intersection of two ICSs be an ICS.

Theorem: 1.22 Let $\mathcal{A} = < A, \lambda>$ and B = < B, μ> be two ICSs in X. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$, be two ICSs in X such that

$$\min\{ A^+(x), B^+(x) \} \geq (\lambda \vee \mu)(x)$$

for all x ∈ X. Then the R-intersection of $\mathcal{A}^*$ and $\mathcal{B}^*$ is an ICS in X.

Proof: Let $\mathcal{A} = < A, \lambda>$ and B = < B, μ> be two ICSs in X. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$, be ICSs in X which satisfy the condition.

Then $A^-(x) \leq ?(x) \leq A^+(x)$ and , $B^-(x) \leq \lambda(x) \leq B^+(x)$ which implies that $(\mu \vee \lambda)(x) \leq (A \cap B)^+(x)$. It follows from the definition of condition that

$$(A \cap B)^-(x) = \min\{A^+(x), B^+(x)\} \leq (\mu \vee \lambda)(x) \leq (A \cap B)^+(x)$$

So that $\mathcal{A}^* \cap_R \mathcal{B}^* = \{<x, (A \cap B)(x), (\mu \vee \lambda)(x) >| x \in X\}$ is an ICS in X.

The following examples show that R-union of ECSs need not be an ICS.

Example: 1.23 Let $\mathcal{A} = < A, \lambda>$ and $\mathcal{B} = < B, \mu>$ be two ECSs. Also let $\mathcal{A}^* = < A, \mu>$, $\mathcal{B}^* = < B, \lambda>$ be two ECSs in I = [0,1] in which A(x) = [0.2, 0.4], λ(x) = 0.7, B(x)=[0.6,0.8] and μ(x)= 0.9 for all x ∈ X. We know that $\mathcal{A}^* \cup_R \mathcal{B}^* = \{< x, B(x), \lambda(x)>| x \in X\}$ and $\lambda(x) \in [B^-(x), B^+(x)]$ for all x ∈ I. Hence $\mathcal{A}^* \cup_R \mathcal{B}^*$ is not an ICS in X.

The following example shows that the R-intersection of ECSs need not be an ECS.

Example: 1.24 Let $\mathcal{A}$ = < A, λ> and B = < B, μ> be two ECSs. Also let $\mathcal{A}^*$ = < A, μ>, $\mathcal{B}^*$ = < B, λ> be two ECSs in I = [0,1] in which A(x) = [0.2, 0.4], λ(x) = 0.1, B(x)=[0.6,0.8] and μ(x)= 0.3 for all x ∈ X. We know that $\mathcal{A}^* \cap_R \mathcal{B}^*$ = {< x, A(x), μ(x)>| x ∈ X} and μ(x) ∉ [A^- (x), A^+(x)] for all x ∈ X. Hence $\mathcal{A}^* \cap_R \mathcal{B}^*$ is not an ECS in X.

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