FUZZY SOFT GAMMA REGULAR SEMIGROUPS

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Abstract:
In this paper, we have discussed about the fuzzy soft \( \Gamma \)-regular and \( \Gamma \)-intra regular semigroups and their properties.

Index Terms: Soft semigroups, soft \( \Gamma \)-ideals, \( \Gamma \)-regular semigroups, \( \Gamma \)-intra regular semigroups, soft \( \Gamma \)-regular semigroups & fuzzy soft \( \Gamma \)-intra regular semigroups.

1. Introduction:

2. Preliminaries:

Definition 2.1 [13]: Let \( S = \{a, b, c, \ldots\} \) and \( \Gamma = \{\alpha, \beta, \gamma, \ldots\} \) be two non-empty sets. Then \( S \) is called a \( \Gamma \)–-semigroup if it satisfies the conditions

(i) \( \alpha a b \in S \),

(ii) \( (\alpha b)\gamma = \alpha b (b \gamma c) \) \( \forall a, b, c \in S \) and \( \alpha, \beta, \gamma \in \Gamma \).

Definition 2.2 [5]: A \( \Gamma \)–semigroup \( S \) is called a regular if for each element \( a \in S \), there exists \( x \in S \) and \( \alpha, \beta \in \Gamma \) such that \( a = \alpha x \beta a \).

Definition 2.3 [8]: Let \( U \) be the universel set, \( E \) be the set of parameters, \( P(U) \) denote the power set of \( U \) and \( A \) be a non-empty subset of \( E \). A pair \((F, A)\) is called a soft set over \( U \), where \( F \) is mapping given by \( F : A \rightarrow P(U) \).

Definition 2.4 [7]: Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \( U \) then \((F, A) \text{ AND } (G, B)\) denoted by \((F, A) \land (G, B)\) is defined as

\[(F, A) \land (G, B) = (H, A \times B) \text{ where } H(\alpha, \beta) = F(\alpha) \cap G(\beta), \forall (\alpha, \beta) \in A \times B\]

Definition 2.5 [7]: Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \( U \) then \((F, A) \text{ OR } (G, B)\) denoted by \((F, A) \lor (G, B)\) is defined as

\[(F, A) \lor (G, B) = (H, A \times B) \text{ where } H(\alpha, \beta) = F(\alpha) \cup G(\beta) \lor (\alpha, \beta) \in A \times B.\]

Definition 2.6 [1]: The extended union of two fuzzy soft sets \((F, A)\) and \((G, B)\) over a common universe \( U \) is fuzzy soft set denoted by \((F, A) \lor_c (G, B)\) defined as

\[(F, A) \lor_c (G, B) = (H, C) \text{ where } C = A \cup B, \forall c \in C.\]
\[ H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \\ F(c) \cap G(c) & \text{if } c \in A \cap B. \end{cases} \]

**Definition 2.7 [1]:** The extended intersection of two fuzzy soft sets \((F, A)\) and \((G, B)\) over a common universe \(U\) is fuzzy soft set denoted by \((F, A) \cap_{e} (G, B)\) defined as \((F, A) \cap_{e} (G, B) = (H, C)\) where \(C = A \cup B, \ \forall c \in C.\)

**Definition 2.8 [1]:** Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \(U\) such that \(A \cap B \neq \phi.\) The restricted intersection of \((F, A)\) and \((G, B)\) is denoted by \((F, A) \cap_{r} (G, B)\) and defined as \((F, A) \cap_{r} (G, B) = (H, C)\) where \(C = A \cap B, \ \forall c \in C.\)

**Definition 2.9 [2]:** The restricted product \((H, C)\) of two fuzzy soft sets \((F, A)\) and \((G, B)\) over a semigroup \(S\) is defined as \((H, C) = (F, A) \cap_{r} (G, B)\) where \(C = A \cap B\) by \(H(c) = F(c) \cap_{r} G(c), \ \forall c \in C.\)

**Definition 2.10 [3]:** A soft set \((F, A)\) is called a soft semigroup over \(S\) if \((F, A) \cap_{r} (G, B) \subseteq (F, A).\) Clearly a soft set \((F, A)\) over a semigroup \(S\) is a soft semigroup if and only if \(\phi \neq F(a)\) is a subsemigroup of \(S, \forall a \in A.\)

**Definition 2.11 [10]:** A soft semigroup \((F, A)\) over a semigroup \(S\) is called a soft regular semigroup if for each \(\alpha \in A, F(\alpha)\) is regular.

**Definition 2.12 [15]:** Let \(X\) be non-empty set. A fuzzy subset \(\mu\) of \(X\) is a function from \(X\) into the closed unit interval \([0, 1]\). The set of all fuzzy subsets of \(X\) is called a fuzzy power set of \(X\) and is denoted by \(FP(X)\).

**Definition 2.13 [9]:** A fuzzy soft set \((\mu, A)\) of a \(\Gamma-\)semigroup \(S\), then \((\mu, A)\) is called a fuzzy soft \(\Gamma-\)subsemigroup of \(S\) if \(\mu_{a}(x \gamma y) \geq \min\{\mu_{a}(x), \mu_{a}(y)\}, \forall a \in A, x, y \in S, \gamma \in \Gamma.\)

**Definition 2.14 [9]:** A fuzzy soft set \((\mu, A)\) of a \(\Gamma-\)semigroup \(S\) is called a fuzzy soft \(\Gamma-\)left(right) ideal of \(S\) if \(\mu_{a}(x \gamma y) \geq \mu_{a}(y), (\mu_{a}(x \gamma y) \geq \mu_{a}(x)), \forall a \in A, x, y \in S\) and \(\gamma \in \Gamma.\)

**Definition 2.15 [9]:** A fuzzy soft \(\Gamma-\)subsemigroup \((\mu, A)\) of a \(\Gamma-\)semigroup \(S\) is called a fuzzy soft \(\Gamma-\)ideal of \(S\) if \(\mu_{a}(x \alpha \beta y) \geq \max\{\mu_{a}(x), \mu_{a}(y)\}, \forall a \in A, x, y, z \in S, \alpha, \beta \in \Gamma.\)

**Definition 2.16 [9]:** A fuzzy soft \(\Gamma-\)subsemigroup \((\mu, A)\) of a \(\Gamma-\)semigroup \(S\) is called a fuzzy soft \(\Gamma-\)bi-ideal of \(S\) if \(\mu_{a}(\alpha \alpha \beta) \geq \min\{\mu_{a}(x), \mu_{a}(y)\}, \forall a \in A, x, y, z \in S, \alpha, \beta \in \Gamma.\)

**Definition 2.17 [9]:** A fuzzy soft \(\Gamma-\)subsemigroup \((\mu, A)\) of a \(\Gamma-\)semigroup \(S\) is called a fuzzy soft \(\Gamma-\)interior ideal of \(S\) if \(\mu_{a}(x \alpha \beta y) \geq \mu_{a}(z), \forall a \in A, x, y, z \in S, \alpha, \beta \in \Gamma.\)

**Definition 2.18 [11]:** A fuzzy soft \((\mu, A)\) over \(S\) is said to be a fuzzy soft quasi ideal of \(S\) if \(\mu(e)\) is a fuzzy quasi ideal of \(S\) \(\forall e \in A.\)

**Definition 2.19 [11]:** A fuzzy soft \((\mu, A)\) over \(S\) is said to be a fuzzy soft generalized bi-ideal of \(S\) if \(\mu(e)\) is a fuzzy generalized bi- ideal of \(S\) \(\forall e \in A.\)

3. Fuzzy Soft Gamma Regular Semigroups:

In this section \(S\) denotes the soft \(\Gamma-\)regular semigroup.
Definition 3.1: A soft $\Gamma$-semigroup $(F, A)$ over a semigroup $S$ is called a soft $\Gamma$-regular semigroup if for each $\alpha, \beta \in A, F(\alpha, \beta)$ is regular.

Example 3.2: $S = \{a_1, a_2, a_3\}$ and $\Gamma = \{\alpha, \beta\}$ where $\alpha, \beta$ is defined on $S$ with the following cayley table:

$\begin{array}{c|cccc} \alpha & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 & a_3 & a_3 \\ a_4 & a_3 & a_3 & a_3 & a_3 \\ \end{array}$

$\begin{array}{c|cccc} \beta & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_1 & a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 & a_3 & a_3 \\ a_4 & a_3 & a_3 & a_3 & a_3 \\ \end{array}$

Table-1

Consider $E = \{a_1, a_2, a_3, a_4\}$ and $F(a_1) = \{a_1, a_3\}$, $F(a_2) = \{a_2, a_3\}$, $F(a_3) = \{a_1, a_2, a_3\}$ $F(a_4) = \{a_2, a_3, a_4\}$. Hence $(F, S)$ is soft $\Gamma$-regular semigroup.

Theorem 3.3: Let $(F, A)$ and $(G, B)$ be two fuzzy soft $\Gamma$-ideal (bi-ideal, interior ideal) over soft $\Gamma$-regular semigroup $S$, then $(F, A) \wedge (G, B)$ is fuzzy soft $\Gamma$-ideal (bi-ideal, interior ideal) over soft $\Gamma$-regular semigroup $S$.

Proof: Let $(F, A)$ and $(G, B)$ be two fuzzy soft $\Gamma$-ideal over soft $\Gamma$-regular semigroup $S$. Now we defined $(F, A) \wedge (G, B) = (H, C)$ where $C = A \times B$ and $H(a, b) = F(a) \cap G(b) \forall (a, b) \in C$.

Consider

$\mu_{H(a,b)}(x\alpha y) = (\mu_{F(a)} \cap \mu_{G(b)}) (x\alpha y)$

$= \min\{\mu_{F(a)}(x\alpha y), \mu_{G(b)}(x\alpha y)\}$

$\geq \min\{\max\{\mu_{F(a)}(x), \mu_{F(a)}(y)\}, \max\{\mu_{G(b)}(x), \mu_{G(b)}(y)\}\}$

$= \max\{\min\{\mu_{F(a)}(x), \mu_{F(a)}(y)\}, \min\{\mu_{G(b)}(x), \mu_{G(b)}(y)\}\}$

$= \max\{\mu_{F(a)} \cap \mu_{G(b)}(x), \mu_{F(a)} \cap \mu_{G(b)}(y)\}$

$= \max\{\mu_{H(a,b)}(x), \mu_{H(a,b)}(y)\}$

Hence $(F, A) \wedge (G, B)$ is fuzzy soft $\Gamma$-ideal over soft $\Gamma$-regular semigroup $S$.

Theorem 3.4: Let $(F, A)$ and $(G, B)$ be two fuzzy soft $\Gamma$-ideal (bi-ideal, interior ideal) over soft $\Gamma$-regular semigroup $S$, then $(F, A) \vee (G, B)$ is fuzzy soft $\Gamma$-ideal (bi-ideal, interior ideal) over soft $\Gamma$-regular semigroup $S$.

Proof: The proof is straightforward.

Theorem 3.5: Let $(F, A)$ and $(G, B)$ be two fuzzy soft $\Gamma$-ideal of $S$, then $(F, A) \cap (G, B)$ is a fuzzy soft $\Gamma$-ideal over $S$.

Proof: Let $(F, A)$ and $(G, B)$ be two fuzzy soft $\Gamma$-ideal over $S$, then $(F, A) \cap (G, B) = (H, C)$ where $C = A \cup B$, $\forall c \in C$

$H(c) = \begin{cases} F(c) & \text{if } c \in A - B \\ G(c) & \text{if } c \in B - A \\ F(c) \cap G(c) & \text{if } c \in A \cap B. \end{cases}$

Let $s, t \in S$ and $\alpha \in \Gamma$.

(i) If $c \in A - B$
\[ \mu_{H(c)}(s \alpha t) = \mu_{F(c)}(s \alpha t) \]
\[ \geq \max\{ \mu_{F(c)}(s), \mu_{F(c)}(t) \} \]
\[ = \max\{ \mu_{H(c)}(s), \mu_{H(c)}(t) \} \]

(ii) If \( c \in B - A \)
\[ \mu_{H(c)}(s \alpha t) = \mu_{G(c)}(s \alpha t) \]
\[ \geq \max\{ \mu_{G(c)}(s), \mu_{G(c)}(t) \} \]
\[ = \max\{ \mu_{H(c)}(s), \mu_{H(c)}(t) \} \]

(iii) If \( c \in A \cap B \) then \( H(c) = \min\{ F(c), G(c) \} = \{ F(c) \cap G(c) \} \)

Now verify that \( H(c)(s \alpha t) \geq \max\{ H(c)(s), H(c)(t) \} \), \( \forall s, t \in S, \alpha \in \Gamma \), \( c \in C \).
Thus \( \mu_{H(c)}(s \alpha t) \geq \max\{ \mu_{H(c)}(s), \mu_{H(c)}(t) \} \). Hence \( (F, A) \cup (G, B) = (H, C) \) is a fuzzy soft \( \Gamma \)–ideal over \( S \).

**Theorem 3.6**: Let \( (F, A) \) and \( (G, B) \) be two fuzzy soft \( \Gamma \)–ideal over \( S \), \( (F, A) \cup (G, B) \) is a fuzzy soft \( \Gamma \)–ideal over \( S \).

**Proof**: The proof is straightforward.

**Theorem 3.7**: Let \( (F, A) \) and \( (G, B) \) be two fuzzy soft sets of \( \Gamma \)–regular semigroup \( S \). \( A_1 \) and \( A_2 \) are two non-empty subsets of \( S \).

(i) \( \chi_{F(c)A_1} \cap \chi_{F(c)A_2} = \chi_{F(c)(A_1 \cap A_2)} \)

(ii) \( \chi_{F(c)A_1} \supseteq \chi_{F(c)A_2} = \chi_{F(c)(A_1 \cap A_2)} \)

**Proof**: Let \( p \in S \), if \( p \in A_1 \cap A_2 \), then \( p \in A_1 \) and \( p \in A_2 \). we have
\( (\chi_{F(c)A_1} \cap \chi_{F(c)A_2})(p) = \min\{ \chi_{F(c)A_1}(p), \chi_{F(c)A_2}(p) \} \)
\[ = \min\{0,0\} \]
\[ = 0 \]
\[ = \chi_{F(c)(A_1 \cap A_2)}(p) \]

Suppose \( p \notin A_1 \cap A_2 \), then \( p \notin A_1 \) and \( p \notin A_2 \)
\( (\chi_{F(c)A_1} \cap \chi_{F(c)A_2})(p) = \min\{ \chi_{F(c)A_1}(p), \chi_{F(c)A_2}(p) \} \)
\[ = \min\{0,0\} \]
\[ = 0 \]
\[ = \chi_{F(c)(A_1 \cap A_2)}(p) \]

Let \( p \in S \) if \( p \in A_1 \Gamma A_2 \), then there exists \( a_1 \in A_1, \gamma \in \Gamma \) and \( a_2 \in A_2 \), such that \( p = a_1 \gamma a_2 \)
\( (\chi_{F(c)A_1} \supseteq \chi_{F(c)A_2})(p) = \sup_{p=cd} \min\{ \chi_{F(c)A_1}(c), \chi_{F(c)A_2}(d) \} \)
\[ \geq \min\{ \chi_{F(c)A_1}(p), \chi_{F(c)A_2}(p) \} \]
\[ = \min\{1,1\} \]
\[ = 1 \]

So \( (\chi_{F(c)A_1} \supseteq \chi_{F(c)A_2})(p) = 1 \) since \( p \in A_1 \Gamma A_2 \). \( \chi_{F(c)(A_1 \Gamma A_2)}(p) = 1 \). Suppose \( p \notin A_1 \Gamma A_2 \), then \( p \notin a_1 \gamma a_2, a_1 \in A_1, \gamma \in \Gamma \) and \( a_2 \in A_2 \).
If $A$ be a soft ideal of $S$, then $\chi_A = \chi_{IA}$.

Hence $\chi_A = \chi_{IA} = \chi_{IA}$.

**Theorem 3.8:** Let $(F, A)$ be a soft subset of a semigroup $S$, $(F, A)$ be a soft $\Gamma$-subsemigroup of $S$ if and only if $\chi_{F(e)}$ is fuzzy soft $\Gamma$-subsemigroup of $S$.

**Proof:** Let $(F, A)$ be a soft $\Gamma$-subsemigroup of $S$.

$\chi_{F(e)}(a) = \begin{cases} 1 & \text{if } a \in F(e) \\ 0 & \text{if } a \notin F(e). \end{cases}$

Let $a, b \in S$, $\gamma \in \Gamma$ if $\chi_{F(e)}(a\gamma b) \leq \min\{\chi_{F(e)}(a), \chi_{F(e)}(b)\}$, then $\chi_{F(e)}(a) = 1$.

$\chi_{F(e)}(b) = 1$ and $\chi_{F(e)}(a\gamma b) = 0$, this implies that $a, b \in F(e)$ since $F(e)$ is a $\Gamma$-subsemigroup of $S$, $amy b \in F(e)$ and hence $\chi_{F(e)}(a\gamma b) = 1$ which is a contradiction. Thus $\chi_{F(e)}(a\gamma b) \leq \min\{\chi_{F(e)}(a), \chi_{F(e)}(b)\}, \forall a, b \in F(e)$ and $e \in A$. Conversely assume that $\chi_{F(e)}$ is soft $\Gamma$-subsemigroup of $S$. Let $a, b \in F(e)$, then $\chi_{F(e)}(a) = 1$, $\chi_{F(e)}(b) = 1$, $\chi_{F(e)}$ fuzzy soft $\Gamma$-subsemigroup. Now $\min\{\chi_{F(e)}(a), \chi_{F(e)}(b)\} = 1 \leq \chi_{F(e)}(a\gamma b)$, this implies that $\chi_{F(e)}(a\gamma b) = 1$ and hence $a, b \in F(e) \forall e \in A$. Therefore $(F, A)$ is a soft $\Gamma$-subsemigroup of $S$.

**Theorem 3.9:** Let $(F, B)$ be a soft $\Gamma$-bi-ideal of $S$ if and only if $\chi_{F(e)}$ (characteristic function) is a fuzzy soft $\Gamma$-bi-ideal of $S$.

**Proof:** Assume that $(F, B)$ be a soft $\Gamma$-bi-ideal of $S$, by theorem (3.7) we have $\chi_{F(e)}(e) \leq S \leq \chi_{F(e)} = \chi_{F(e)}(B) = \chi_{F(e)}(B)$. Hence $\chi_{F(e)}(e)$ is a fuzzy soft $\Gamma$-bi-ideal of $S$.

Conversely assume that $\chi_{F(e)}(e)$ is a fuzzy soft $\Gamma$-bi-ideal of $S$, $B \subseteq S$, by theorem (3.8), it is clear that $B$ is a soft $\Gamma$-subsemigroup of $S$.

Let $p \in S$ such that $p \in B\Gamma\Sigma B$, then $\chi_{F(e)}(e)$ is a fuzzy soft $\Gamma$-bi-ideal, we have $\chi_{F(e)}(e)(p) = (\chi_{F(e)}(e) \leq S \leq \chi_{F(e)}(B))(p) = 1$ which implies that $p \in B$ and hence $B\Gamma\Sigma B \subseteq B$. Hence $(F, B)$ is a soft $\Gamma$-bi-ideal of $S$.

**Theorem 3.10:** Let $(F, A)$ be a soft $\Gamma$-ideal of $S$ if and only if $\chi_{F(e)}(e)$ (characteristic function) is a fuzzy soft $\Gamma$-ideal of $S$.

**Proof:** The proof is straightforward.

The following theorem is relation between soft set and fuzzy set.

**Theorem 3.11:** Let $(Q, A)$ be a soft subset of a $\Gamma$-semigroup $S$, then $(Q, A)$ is a soft $\Gamma$-quasi ideal of $S$ if and only if $\chi_Q$ (characteristic function) is fuzzy soft $\Gamma$-quasi ideal of $S$.

**Proof:** Suppose $(Q, A)$ is a soft $\Gamma$-quasi ideal of $S$ and $\chi_{F(e)}$ be the characteristic function of $S$, let $x \in S$. If $x \in (Q, A)$ then $((\chi_Q \leq \chi_{F(e)}) \cap (\chi_{F(e)} \leq \chi_Q))(x) \leq 1 = f_Q(x)$. If $x \notin (Q, A)$ then $x \notin (Q, A)(S, E) \cap (S, E)(Q, A) \subseteq (Q, A)$.

**Case(i):** Let $x \notin (Q, A)(S, E), x \notin (S, E)(Q, A)$. If $x = a\gamma b$ then $a \notin (Q, A)$. Then
\[(\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q)) (x) = \min \{ \sup_{x=\alpha} \{ \min \{ \chi_Q (a), \chi_{F(e)} (b) \} \} \text{ if } x \neq (Q,A) \}\text{ and } a, b \in S.\]

Therefore \((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q) \subseteq \chi_Q \).

**Case (ii):** Let \(x \in (Q,A) \Gamma(S,E) \), \(x \notin (S,E) \Gamma(Q,A) \). If \(x = u \gamma_2 v\), then \(v \notin (Q,A) \). Then

\[((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q)) (x) = \min \{ \sup_{x=\alpha} \{ \min \{ \chi_Q (a), \chi_{F(e)} (b) \} \} \}, \sup \{ \min \{ \chi_{F(e)} (u), \chi_Q (v) \} \}\]

\[= 0 = \chi_Q (x).\]

Therefore \((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q) \subseteq \chi_Q \).

**Case (iii):** Let \(x \notin (Q,A) \Gamma(S,E), \ x \notin (S,E) \Gamma(Q,A) \). If \(x = a \gamma_2 b\) then \(a \notin (Q,A) \), then and if \(x = u \gamma_2 v\), then \(v \notin (Q,A) \). Then

\[((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q)) (x) = \min \{ \sup_{x=\alpha} \{ \min \{ \chi_Q (a), \chi_{F(e)} (b) \} \} \}, \sup \{ \min \{ \chi_{F(e)} (u), \chi_Q (v) \} \}\]

\[= 0 = \chi_Q (x).\]

Therefore \((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q) \subseteq \chi_Q \). Hence \(\chi_Q \) is a fuzzy soft \(\Gamma\) - quasi ideal of \(S\). Conversely, suppose that \(\chi_Q \) is a fuzzy soft \(\Gamma\) - quasi ideal of \(S\).

Let \(x \in (\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q) \) then there exists \(s,t \in S, \ y,z \in (Q,A) \) and \(\alpha, \beta \in \Gamma\) such that \(x = y \alpha \beta = t \beta \). Consider

\((\chi_Q \circlearrowleft \chi_{F(e)})(x) = \sup_{x=\alpha} \{ \min \{ \chi_Q (a), \chi_{F(e)} (b) \} \}\)

\[\geq \min \{ \chi_Q (y), \chi_{F(e)} (s) \}\]

\[= \min \{1,1\}\]

\[= 1\]

Similarly, \((\chi_{F(e)} \circlearrowleft \chi_Q)(x) = 1\). Since \((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q) \subseteq \chi_Q \).

Consider

\(\chi_Q (x) \geq ((\chi_Q \circlearrowleft \chi_{F(e)}) \cap (\chi_{F(e)} \circlearrowleft \chi_Q)) (x)\)

\[= \min \{(\chi_Q \circlearrowleft \chi_{F(e)})(x), (\chi_{F(e)} \circlearrowleft \chi_Q)(x)\}\]

\[= \min \{1,1\} = 1\]

Thus \(x \in (Q,A) \) and hence \((Q,A) \Gamma(S,E) \cap (S,E) \Gamma(Q,A) \subseteq (Q,A)\). Therefore \((Q,A)\) is a soft \(\Gamma\) - quasi ideal of \(S\).

**Theorem 3.12:** The following conditions are equivalent

(i) Every soft \(\Gamma\) -bi-ideal is a soft \(\Gamma\) -ideal of \(S\).

(ii) Every fuzzy soft \(\Gamma\) -bi-ideal of \(S\) is a fuzzy soft \(\Gamma\) -ideal of \(S\).

**Proof:** Assume that condition (i) holds, let \(\mu_{F(e)} \) be any fuzzy soft \(\Gamma\) -ideal of \(S\). Let \(\mu_{F(e)} \) be any fuzzy soft \(\Gamma\) -bi-ideal of \(S\) and \(p,q \in S\), since the set \((F,A) \circlearrowleft (S,E) \circlearrowleft (F,A) \) is a soft \(\Gamma\) -bi-ideal of \(S\), by the assumption is soft \(\Gamma\) -right ideal of \(S\) is soft \(\Gamma\) -regular, we have

\[p \Gamma q \in (p \alpha F(\alpha, \beta) \beta p) \Gamma q \subseteq p \alpha F(\alpha, \beta) \beta p,\] there exists \(x \in S\) such that \(p \Gamma q = p \alpha \beta \beta p\), since \(\mu_{F(e)} \) is a fuzzy soft \(\Gamma\) -bi-ideal of \(S\), \forall e \in A.

Consider
\[ \mu_{F(x)}(p \Gamma q) = \mu_{F(x)}(p) \alpha \beta p \]
\[ \mu_{F(x)}(p \Gamma q) = \mu_{F(x)}(p) \alpha \beta p \Gamma q \]
\[ \geq \min(\mu_{F(x)}(p), \mu_{F(x)}(p)) \]
\[ = \mu_{F(x)}(p). \]

Hence \( \mu_{F(x)} \) is a fuzzy soft \( \Gamma \)-right ideal of \( S \). Similarly \( \mu_{F(x)} \) is a fuzzy soft \( \Gamma \)-left ideal of \( S \). Therefore \( \mu_{F(x)} \) is a fuzzy soft \( \Gamma \)-ideal of \( S \). Hence \( (i) \Rightarrow (ii) \).

Conversely assume that \( (ii) \) holds. Let \( (F, A) \) be a soft \( \Gamma \)-ideal of \( S \), by theorem (3.9), the characteristic function \( \chi_{F(x)}A \) is a fuzzy soft \( \Gamma \)-ideal of \( S \). Hence by assumption \( \chi_{F(x)}A \) is a fuzzy soft \( \Gamma \)-ideal of \( S \). Thus, by theorem (3.10), \( \chi_{F(x)}A \) is a fuzzy soft \( \Gamma \)-ideal of \( S \). Hence \( (ii) \Rightarrow (i) \).

The following examples shows that fuzzy soft \( \Gamma \)-ideal and fuzzy soft \( \Gamma \)-bi-ideal of \( S \).

**Examples 3.13:** Let \( S = \{a_1, a_2, a_3, a_4\} \) and \( \Gamma = \{\alpha, \beta\} \) in the table 1.

Let \( E = \{a_1, a_2, a_3, a_4\} \), \( A = \{u_1, u_3\} \) then \( (F, A) \) is a fuzzy soft set defined as,

\[ \mu_{F(u_1)} = \{(a_1, 0.9), (a_2, 0.3), (a_3, 0.7), (a_4, 0.3)\} \]
\[ \mu_{F(u_3)} = \{(a_1, 0.8), (a_2, 0.1), (a_3, 0.5), (a_4, 0.1)\} \]

Hence \( (F, A) \) is a fuzzy soft \( \Gamma \)-bi-ideal and fuzzy soft \( \Gamma \)-ideal over \( S \).

**Theorem 3.14:** The following conditions are equivalent.

(i) \( (F, A) \) is a fuzzy soft \( \Gamma \)-ideal of \( S \).

(ii) \( (F, A) \) is a fuzzy soft \( \Gamma \)-interior ideal of \( S \).

**Proof:** Let \( \mu_{F(x)} \) is a fuzzy soft \( \Gamma \)-ideal of \( S \), \( \forall e \in A \).

We have \( \mu_{F(x)}(p \alpha q \beta r) \geq \mu_{F(x)}(q \beta r) \), since \( \mu_{F(x)} \) is a \( \Gamma \)-left ideal of \( S \).

\[ \geq \mu_{F(x)}(q), \text{ since } \mu_{F(x)} \text{ is a } \Gamma \text{-right ideal of } S. \]

Hence \( \mu_{F(x)}(p \alpha q \beta r) \geq \mu_{F(x)}(q) \) \( \forall p, q, r \in S \) and \( \alpha, \beta \in \Gamma \).

Conversely assume that \( \mu_{F(x)} \) is a fuzzy soft \( \Gamma \)-interior ideal of \( S \). Let \( p, q \in S \), since \( S \) is a soft \( \Gamma \)-regular semigroups, there exists \( x, y \in S \), such that \( p = p \alpha x \beta p \) and \( q = q \alpha y \beta q \) and \( \alpha, \beta \in \Gamma \). Thus we have

\[ \mu_{F(x)}(p \Gamma q) = \mu_{F(x)}((p \alpha x \beta p) \Gamma q) \]
\[ = \mu_{F(x)}((p \alpha x \beta p) \Gamma q) \]
\[ \geq \mu_{F(x)}(p) \]

and

\[ \mu_{F(x)}(p \Gamma q) = \mu_{F(x)}(q \alpha y \beta q) \]
\[ = \mu_{F(x)}(q \alpha y \beta q) \]
\[ \geq \mu_{F(x)}(q) \]

Hence proved.

The following example shows that fuzzy soft \( \Gamma \)-ideal and \( \Gamma \)-interior ideal of \( S \).

**Examples 3.15:** Let \( S = \{a_1, a_2, a_3, a_4\} \) and \( \Gamma = \{\alpha, \beta\} \) in the table 1.

Let \( E = \{a_1, a_2, a_3, a_4\} \), \( A = \{u_1, u_3\} \) then \( (F, A) \) is a fuzzy soft set defined as,

\[ \mu_{F(u_1)} = \{(a_1, 0.8), (a_2, 0.2), (a_3, 0.4), (a_4, 0.2)\} \]
\[ \mu_{F(u_3)} = \{(a_1, 0.7), (a_2, 0.3), (a_3, 0.6), (a_4, 0.3)\} \]

Hence \( (F, A) \) is a fuzzy soft \( \Gamma \)-interior ideal and fuzzy soft \( \Gamma \)-ideal over \( S \).
Theorem 3.16: Every fuzzy soft $\Gamma$-quasi ideals are fuzzy soft $\Gamma$-bi-ideal of $S$. 

Proof: Let $\mu_{F(e)}$ is a fuzzy soft $\Gamma$-quasi ideal of $S$. It is sufficient to prove that 

$$\mu_{F(e)}(p \alpha q \beta r) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\} \quad \forall \ p, q, r \in S, \ e \in A$$

since $\mu_{F(e)}$ is a fuzzy soft $\Gamma$-quasi ideal of $S$. Consider 

$$\mu_{F(e)}(p \alpha q \beta r) = \left( \mu_{F(e)}(p) \cap \mu_{F(e)}(q) \cap \mu_{F(e)}(r) \right)$$

Then since $\mu_{F(e)}(p \alpha q \beta r) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\}$, which is impossible by $\mu_{F(e)} \geq \mu_{F(e)}(e)$, so equation $(i)$ is satisfied. Since $\mu_{F(e)}(p) \geq \mu_{F(e)}(r)$, 

$$\mu_{F(e)}(p) = \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\} = \mu_{F(e)}(r)$$

Hence $\mu_{F(e)}(p \alpha q \beta r) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\}$. 

Theorem 3.17: In a soft $\Gamma$-regular semigroup $S$, then fuzzy soft $\Gamma$-quasi-ideals and fuzzy soft $\Gamma$-bi-ideals coincide. 

Proof: It is remains to prove that every fuzzy soft $\Gamma$-bi-ideals are fuzzy soft $\Gamma$-quasi-ideals if $\mu_{F(e)}$ be a fuzzy soft $\Gamma$-bi-ideal of $S$, then 

$$\mu_{F(e)}(p \alpha q \beta r) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\}$$

(i) Let $\ p \in S$, suppose that $(\mu_{F(e)}(p \alpha q \beta r)) \leq \mu_{F(e)}(p) \forall e \in A$. Consider 

$$(\mu_{F(e)}(p \alpha q \beta r)) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\}$$

Now suppose that $(\mu_{F(e)}(p \alpha q \beta r)) \geq \mu_{F(e)}(p)$, then there exists $x, y \in S, p = x \gamma y$ such that 

$$\min\{\mu_{F(e)}(x), \mu_{F(e)}(y)\} \geq \mu_{F(e)}(p)$$

and 

$$\mu_{F(e)}(x) \geq \mu_{F(e)}(p)$$

Now we prove that $(\mu_{F(e)}(p \alpha q \beta r)) \leq \mu_{F(e)}(p)$, so equation $(i)$ is satisfied. Since 

$$(\mu_{F(e)}(p \alpha q \beta r)) = \min\{\mu_{F(e)}(p), \mu_{F(e)}(r)\}$$

Prove that $\min\{\mu_{F(e)}(q), \mu_{F(e)}(r)\} \leq \mu_{F(e)}(p) \forall p, q, r \in S$, since $S$ is soft $\Gamma$-regular, there exists $u \in S$ such that $p = p \alpha u \beta p = x \gamma u \alpha \beta y \gamma r$. Then since $\mu_{F(e)}$ is a fuzzy soft $\Gamma$-bi-ideal, we have $\mu_{F(e)}(p) = \mu_{F(e)}(x \gamma u \alpha \beta y \gamma r) \geq \min\{\mu_{F(e)}(x), \mu_{F(e)}(r)\}$. 

If $\min\{\mu_{F(e)}(x), \mu_{F(e)}(r)\} = \mu_{F(e)}(x)$, then $\mu_{F(e)}(x) \leq \mu_{F(e)}(p)$, which is impossible by equation $(ii)$, thus 

$$\min\{\mu_{F(e)}(x), \mu_{F(e)}(r)\} = \mu_{F(e)}(r)$$

and 

$$\mu_{F(e)}(p) \geq \mu_{F(e)}(r) = \min\{\mu_{F(e)}(q), \mu_{F(e)}(r)\}$$

$\forall e \in A$. 

Theorem 3.18: The following conditions are equivalent 

$(i)$ $(F, A)$ is $\Gamma$-regular.
(ii) \((B, \lambda_1) \cap_R (L, \lambda_2) \subseteq (B, \lambda_1) \circlearrowleft (L, \lambda_2)\) for every soft \(\Gamma\)-bi-ideal \((B, \lambda_i)\) and every soft \(\Gamma\)-left ideal \((L, \lambda_2)\) of \(S\).

**Proof:**
(i) \(\Rightarrow\) (ii) By definition \((B, \lambda_1) \circlearrowleft (L, \lambda_2) = (K_1, \lambda_1 \cap \lambda_2)\) where \(K_1\) is a function 
\(K_1 : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\alpha, \beta)), K_1(\alpha, \beta) = B(\alpha, \beta) \Gamma L(\beta) \ \forall \ \alpha, \beta \in \lambda_1 \cap \lambda_2\).

Now \((B, \lambda_1) \cap_R (L, \lambda_2) = (K_2, \lambda_1 \cap_R \lambda_2)\) where \(K_2\) is a function \(\lambda_1 \cap_R \lambda_2\) to \(P(S)\) such that 
\(K_2(\alpha, \beta) = B(\alpha, \beta) \cap L(\beta) \ \forall \ \alpha, \beta \in \lambda_1 \cap \lambda_2\).

We suppose that \(p \in B(\alpha, \beta) \cap L(\beta)\), since \(p \in F(\alpha, \beta)\) and \(F(\alpha, \beta)\) is regular, there exists \(q \in S\) such that \(p = \rho a q \beta p\).

\(F(\alpha, \beta)\) to \(PF(\alpha, \beta)\) defined by \(L(p) = F(\alpha, \beta) \cap L(\beta)\).

Then \((B, F(\alpha, \beta))\) is a soft \(\Gamma\)-bi-ideal and \((L, F(\alpha, \beta))\) is a soft \(\Gamma\)-left ideal over \(F(\alpha, \beta)\) by hypothesis. If \(p \in B(\rho a q \beta p) \cap L(p) = (B(p) \Gamma L(p) = p \alpha F(\alpha, \beta) \beta p \gamma F(\alpha, \beta) \beta p \subseteq p \alpha F(\alpha, \beta) \beta p\).

Therefore \((F, A)\) is soft \(\Gamma\)-regular.

**Theorem 3.19:** Every fuzzy soft \(\Gamma\)-generalized bi-ideal is a fuzzy soft \(\Gamma\)-bi-ideal of \(S\).

**Proof:** Let \(\mu_{F(e)}\) be any fuzzy soft \(\Gamma\)-generalized bi-ideal of \(S\) and let \(q \in S\), since \(S\) is a soft \(\Gamma\)-regular there exists \(x \in S\), such that \(q = \rho a x \beta q\).

\[\mu_{F(e)}(p \Gamma q) = \mu_{F(e)}(p \Gamma (q \alpha x \beta p)) = \mu_{F(e)}(p \Gamma (q \alpha x) \beta q) \geq \min\{\mu_{F(e)}(p), \mu_{F(e)}(q)\}.\]

This implies that \(\mu_{F(e)}\) is a fuzzy soft \(\Gamma\)-subsemigroup of \(S\) and so \(\mu_{F(e)}\) is a fuzzy soft \(\Gamma\)-bi-ideal of \(S\).

**Theorem 3.20:** The following conditions are equivalent
(i) \((F, A)\) is \(\Gamma\)-regular.
(ii) \((B, \lambda_1) \cap_R (L, \lambda_2) \subseteq (B, \lambda_1) \circlearrowleft (L, \lambda_2)\) for every fuzzy soft \(\Gamma\)-generalized bi-ideal \((B, \lambda_1)\) and every fuzzy soft \(\Gamma\)-left ideal \((L, \lambda_2)\) of \(S\).
(iii) \((B, \lambda_1) \cap_R (L, \lambda_2) \subseteq (B, \lambda_1) \circlearrowleft (L, \lambda_2)\) for every fuzzy soft \(\Gamma\)-bi-ideal \((B, \lambda_1)\) and every fuzzy soft \(\Gamma\)-left ideal \((L, \lambda_2)\) of \(S\).

**Proof:** (i) \(\Rightarrow\) (ii). By definition \((B, \lambda_1) \circlearrowleft (L, \lambda_2) = (N, \lambda_1 \cap \lambda_2)\) where \(N\) is a function 
\(N : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\alpha, \beta)), N(\alpha, \beta) = L(\alpha) \Gamma B(\alpha, \beta) \ \forall \ \alpha, \beta \in (\lambda_1 \cap \lambda_2)\).

Let \((B, \lambda_1)\) be a fuzzy soft generalized \(\Gamma\)-bi-ideal and \((L, \lambda_2)\) be fuzzy soft \(\Gamma\)-left ideal of \(S\). Since \((F, A)\) is soft \(\Gamma\)-regular, let \(a \in F(\alpha, \beta)\) there exists \(x \in S\) such that \(a = \rho a x \beta a\).

Consider 
\[(\lambda_1 \circlearrowleft \lambda_2) = \sup \min \{\lambda_1(p), \lambda_2(q)\} \geq \min \{\lambda_1(a), \lambda_2(x \beta a)\} = \min \{\lambda_1(a), \lambda_2(a)\} = (\lambda_1(a) \cap \lambda_2(a))(a)\]

Which implies that \((B, \lambda_1) \cap_R (L, \lambda_2) \subseteq (B, \lambda_1) \circlearrowleft (L, \lambda_2)\).

(ii) \(\Rightarrow\) (iii). By theorem 3.19.
(iii) $\Rightarrow$ (i). Let $(B, \lambda_1)$ be a fuzzy soft $\Gamma$-bi-ideal and $(L, \lambda_2)$ be fuzzy soft $\Gamma$-ideal of $S$, let $a \in (B, \lambda_1) \cap R (L, \lambda_2)$, by theorem (3.10) and theorem (3.9), $\chi_{F(e)l}$ is a fuzzy soft $\Gamma$-left ideal and $\chi_{F(e)b}$ is a fuzzy soft $\Gamma$-bi-ideal of $S$, and $(B, \lambda_1) \cap (L, \lambda_2) = (N, \lambda_1 \cap \lambda_2)$ where $N : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\alpha, \beta))$ and $N(\alpha, \beta) = L(\alpha) \Gamma B(\alpha, \beta) \quad \forall \alpha, \beta \in (\lambda_1 \cap \lambda_2).$ Then by hypothesis $\chi_{F(e)b} \cap \chi_{F(e)l} \subseteq \chi_{F(e)b} \cap \chi_{F(e)l} \subseteq \chi_{F(e)l}$. We have

$$(\chi_{F(e)b} \cap \chi_{F(e)l})(p) = \min \{\chi_{F(e)b}(p), \chi_{F(e)l}(p)\}$$

$$= \min \{1, 1\} = 1.$$ 

$\forall a \in (B, \lambda_1) \cap R (L, \lambda_2)$, since $\chi_{F(e)b} \cap \chi_{F(e)l}$ is a fuzzy soft subset of $S$, we have

$$(\chi_{F(e)b} \cap \chi_{F(e)l})(a) \leq 1 \quad \forall a \in S.$$ 

Consider

$$(\chi_{F(e)b} \cap \chi_{F(e)l})(a) = \sup_{a \in p \in q} \min \{\chi_{F(e)b}(p), \chi_{F(e)l}(q)\}$$

$$\geq \min \{\chi_{F(e)b}(a), \chi_{F(e)l}(a)\}$$

$$= \min \{1, 1\} = 1.$$ 

Hence $\chi_{F(e)b} \cap \chi_{F(e)l} = 1$ thus $1 = \sup_{a \in p \in q} \min \{\chi_{F(e)b}(p), \chi_{F(e)l}(q)\}$, which implies that

$\chi_{F(e)b}(p) = 1$ and $\chi_{F(e)l}(q) = 1$. It follows that $p \in (B, \lambda_1)$, $q \in (L, \lambda_2)$ then

$a = p \Gamma q \in (B, \lambda_1) \Gamma (L, \lambda_2).$ Therefore $(B, \lambda_1) \cap R (L, \lambda_2) \subseteq (B, \lambda_1) \cap (L, \lambda_2)$, by theorem (3.18), hence $(F, A)$ is soft $\Gamma$-regular semigroup.

**Theorem 3.21:** The following conditions are equivalent

(i) $(F, A)$ is left $\Gamma$-regular.

(ii) $(I, \lambda_1) \cap_R (B, \lambda_2) \subseteq (I, \lambda_1) \cap (B, \lambda_2)$ for every fuzzy soft $\Gamma$-ideal $(I, \lambda_1)$ and fuzzy soft $\Gamma$-bi-ideal $(B, \lambda_2)$ of $S$.

**Proof:** (i) $\Rightarrow$ (ii) By definition $(I, \lambda_1) \cap (B, \lambda_2) = (K, \lambda_1 \cap \lambda_2)$ where $K$ is a function $K : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\alpha, \beta))$, $K(\alpha) = I(\alpha, \beta) \Gamma B(\alpha, \beta), \forall \alpha, \beta \in \lambda_1 \cap \lambda_2.$ Let $(F, A)$ be soft $\Gamma$-left regular and $a \in F(\alpha, \beta)$, then there exists $p \in S$ and $\alpha, \beta \in \Gamma$ such that $a = p \alpha a \beta a$.

Let $(I, \lambda_1)$ be fuzzy soft $\Gamma$-ideal and $(B, \lambda_2)$ be fuzzy soft $\Gamma$-bi-ideal of $S$. Consider

$$(\lambda_1 \cap \lambda_2)(a) = \sup_{a \in p \in q} \min \{\lambda_1(p), \lambda_2(q)\}$$

$$\geq \min \{\lambda_1(p \alpha a), \lambda_2(a)\}$$

$$\geq \min \{\lambda_1(a), \lambda_2(a)\}$$

$$(\lambda_1 \cap \lambda_2)(a)$$

Hence $(I, \lambda_1) \cap_R (B, \lambda_2) \subseteq (I, \lambda_1) \cap (B, \lambda_2)$.

(ii) $\Rightarrow$ (i) suppose $(I, \lambda_1)$ and $(B, \lambda_2)$ are fuzzy soft $\Gamma$-ideal and fuzzy soft $\Gamma$-bi-ideal of $S$ such that $(I, \lambda_1) \cap_R (B, \lambda_2) \subseteq (I, \lambda_1) \cap (B, \lambda_2)$. Let $I$ be any soft $\Gamma$-ideal and $B$ be any soft $\Gamma$-bi-ideal of $S$, $p \in I \cap B$, by theorem (3.10) $\chi_{F(e)l}$ is fuzzy soft $\Gamma$-ideal of $S$ and by theorem (3.9) $\chi_{F(e)b}$ is fuzzy soft $\Gamma$-bi-ideal of $S$. Now by theorem (3.7) we have $\chi_{F(e)l} \cap \chi_{F(e)b}$ and hence
1 = (χ_{F(e)r▽B})(p)
= (χ_{F(e)r▽B})\cap (χ_{F(e)B})(p)
≤ (χ_{F(e)r▽B})(p)
= χ_{F(e)P▽B}(p) \text{ by theorem (3.7)}

It follows that \( p \in (I,λ_1) \supseteq (B,λ_2) \) and hence \((I,λ_1) \cap_R (B,λ_2) \subseteq (I,λ_1) \supseteq (B,λ_2)\).

Hence \((F,A)\) is soft left \( \Gamma \) regular.

4. Fuzzy Soft Gamma Intra Regular Semigroups:

In this section \( S \) denotes the soft \( \Gamma - \) intra regular semigroup.

**Definition 4.1:** A soft \( \Gamma - \) semigroup \((F,A)\) over a semigroup \( S \) is called a soft \( \Gamma - \) intra regular semigroup if for each \( \alpha, \beta, \gamma \in A, F(\alpha, \beta, \gamma) \) is intra regular.

**Example 4.2:** \( S = \{a_1, a_2, a_3, a_4\} \), \( \Gamma = \{\alpha, \beta, \gamma\} \) where \( \alpha, \beta, \gamma \) is defined on \( S \) with the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( a_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
<td>( a_1 )</td>
</tr>
<tr>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_4 )</td>
</tr>
<tr>
<td>( a_3 )</td>
<td>( a_1 )</td>
<td>( a_3 )</td>
<td>( a_3 )</td>
<td>( a_3 )</td>
</tr>
<tr>
<td>( a_4 )</td>
<td>( a_1 )</td>
<td>( a_2 )</td>
<td>( a_3 )</td>
<td>( a_4 )</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|cccc} \alpha & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_3 & a_4 \\ a_3 & a_1 & a_3 & a_3 & a_3 \\ a_4 & a_1 & a_2 & a_3 & a_4 \\ \end{array}
\begin{array}{c|cccc} \beta & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_3 & a_3 \\ a_3 & a_1 & a_3 & a_3 & a_3 \\ a_4 & a_1 & a_2 & a_3 & a_4 \\ \end{array}
\begin{array}{c|cccc} \gamma & a_1 & a_2 & a_3 & a_4 \\ \hline a_1 & a_1 & a_1 & a_1 & a_1 \\ a_2 & a_2 & a_2 & a_3 & a_3 \\ a_3 & a_1 & a_3 & a_3 & a_3 \\ a_4 & a_1 & a_2 & a_3 & a_4 \\ \end{array}
\]

Consider \( E = \{a_1, a_2, a_3, a_4\} \) and \( F(a_1) = \{a_1, a_3\}, F(a_2) = \{a_1, a_2, a_3\}, F(a_3) = \{a_2, a_3, a_4\} \)

\( F(a_4) = \{a_1, a_3, a_4\} \) Hence \((F,S)\) is soft \( \Gamma - \) intra regular semigroup.

**Theorem 4.3:** Let \((F,A)\) and \((G,B)\) be two fuzzy soft \( \Gamma - \) ideal (bi-ideal, interior ideal) over soft \( \Gamma - \) intra regular semigroup \( S \), then \((F,A) \wedge (G,B)\) is fuzzy soft \( \Gamma - \) ideal (bi-ideal, interior ideal) over soft \( \Gamma - \) intra regular semigroup \( S \).

**Proof:** The proof is Straightforward.

**Theorem 4.4:** Let \((F,A)\) and \((G,B)\) be two fuzzy soft \( \Gamma - \) ideal (bi-ideal, interior ideal) over soft \( \Gamma - \) intra regular semigroup \( S \), then \((F,A) \vee (G,B)\) is fuzzy soft \( \Gamma - \) ideal (bi-ideal, interior ideal) over soft \( \Gamma - \) intra regular semigroup \( S \).

**Proof:** The proof is Straightforward.

**Theorem 4.5:** The following conditions are equivalent.

(i) \((F,A)\) is a fuzzy soft \( \Gamma - \) ideal of \( S \).

(ii) \((F,A)\) is a fuzzy soft \( \Gamma - \) interior ideal of \( S \).

**Proof:** Let \( \mu_{F(e)} \) is a fuzzy soft \( \Gamma - \) ideal of \( S, \forall e \in A \).

We have \( \mu_{F(e)}(pαβr) \geq \mu_{F(e)}(qβr) \), since \( \mu_{F(e)} \) is a \( \Gamma - \) left ideal of \( S \).

\( \geq \mu_{F(e)}(q) \), since \( \mu_{F(e)} \) is a \( \Gamma - \) right ideal of \( S \).

Hence \( \mu_{F(e)}(pαβr) \geq \mu_{F(e)}(q) \) \( \forall p, q, r \in S \) and \( α, β \in Γ \).

Conversely assume that \( \mu_{F(e)} \) is a fuzzy soft \( \Gamma - \) interior ideal of \( S \). Let \( p, q \in S \), since \( S \) is a soft \( \Gamma - \) intra regular semigroups, there exists \( x, y, u, v \in S \), such that \( p = xαpβpyγ \) and
\[ q = u \alpha q \beta q \gamma \nu \text{ and } \alpha, \beta, \gamma \in \Gamma. \text{ Thus we have} \]
\[ \mu_{F(e)}(p \Gamma q) = \mu_{F(e)}((x \alpha q \beta p \gamma \nu) \Gamma q) \]
\[ = \mu_{F(e)}((x \alpha p \beta p q \gamma \nu)) \]
\[ \geq \mu_{F(e)}(p) \]
\[ \mu_{F(e)}(p \Gamma q) = \mu_{F(e)}(p \Gamma (u \alpha q \beta q \nu)) \]
\[ \text{and} \]
\[ = \mu_{F(e)}(p \Gamma u \alpha q \beta q \nu) \]
\[ \geq \mu_{F(e)}(q) \]

Hence proved.

**Theorem 4.6:** The following conditions are equivalent (i) \((F, A)\) is \(\Gamma\)-intra regular.
(ii) \((L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)\) for every soft \(\Gamma\)-left ideal \((L, \lambda_1)\) and every soft \(\Gamma\)-right ideal \((R, \lambda_2)\) of \(S\).

**Proof:** (i) \(\Rightarrow\) (ii) By definition \((L, \lambda_1) \supseteq (R, \lambda_2) = (K_1, \lambda_1 \cap \lambda_2)\) where \(K_1\) is a function \(K_1 : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\alpha, \beta)), K_1(\Gamma) = L(\alpha) \Gamma \Gamma (\gamma) \forall \gamma \in \lambda_1 \cap \lambda_2\). We have \((L, \lambda_1) \cap_R (R, \lambda_2) = (K_2, \lambda_1 \cap_R \lambda_2)\) where \(K_2\) is a function \(\lambda_1 \cap_R \lambda_2 \rightarrow P(S)\) such that \(K_1, \lambda_1 = K_2, \lambda_1 \subseteq (R, \lambda_2)\).

(iii) \(\Rightarrow\) (i). Suppose that \(a \in L(\alpha) \cap R(\beta)\), then there exists \(p, q \in S\) such that \(a = p \alpha a \beta q \gamma \). Now \(p \alpha a \in L(\alpha), a \beta q \gamma \in R(\gamma)\). This shows that \((L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)\).

Thus we have \(a \in L(\alpha) \cap R(\beta)\), \(F(\alpha, \beta)\) is intra regular, there exists \(p, q \in S\) such that \(a = p \alpha a \beta q \gamma \). Now \(p \alpha a \in L(\alpha), a \beta q \gamma \in R(\gamma)\). This shows that \((L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)\).

(iii) \(\Rightarrow\) (i). Suppose that \(\lambda_1 = \lambda_2 = F(\Gamma)\) and \(N\) is a function from \(F(\Gamma)\) to \(P(F(\Gamma))\) defined by \(L(a) = F(\alpha) \alpha a \beta \gamma \), \(\forall a \in F(\Gamma)\) and \(R\) is a function from \(F(\Gamma)\) to \(P(F(\Gamma))\) defined by \(R(a) = a \beta F(\gamma) \), \(\forall a \in F(\Gamma)\). Then \((L, F(\Gamma))\) is a soft \(\Gamma\)-left ideal and \((R, F(\Gamma))\) is a soft \(\Gamma\)-right ideal over \(F(\Gamma)\) by hypothesis.

\[ a \in L(\alpha) \cap_R (R, \lambda_2) = L(\alpha) \Gamma R(\alpha) = F(\alpha) \alpha a \beta q \gamma F(\gamma). \]

Therefore \((F, A)\) is soft \(\Gamma\)-intra regular.

**Theorem 4.7:** The following conditions are equivalent (i) \((F, A)\) is left \(\Gamma\)-intra regular.
(ii) \((L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)\) for every fuzzy soft \(\Gamma\)-left ideal \((L, \lambda_1)\) and fuzzy soft \(\Gamma\)-right ideal \((R, \lambda_2)\) of \(S\).

**Proof:** (i) \(\Rightarrow\) (ii) By definition \((L, \lambda_1) \supseteq (R, \lambda_2) = (M_1, \lambda_1 \cap \lambda_2)\) where \(M_1\) is a function \(M_1 : (\lambda_1 \cap \lambda_2) \rightarrow P(F(\Gamma)), M_1(\Gamma) = L(\alpha) \Gamma \Gamma (\gamma) \forall \gamma \in \lambda_1 \cap \lambda_2\).

\((L, \lambda_1) \cap_R (R, \lambda_2) = (M_2, \lambda_1 \cap_R \lambda_2)\) where \(M_2\) is a function \((\lambda_1 \cap_R \lambda_2) \rightarrow P(S)\) such that \(M_2(\Gamma) = L(\alpha) \cap_R R(\gamma) \forall \gamma \in \lambda_1 \cap \lambda_2\). Let \((F, A)\) be soft \(\Gamma\)-intra regular and \(a \in F(\Gamma)\), then there exists \(p, q \in S\) and \(a, \beta, \gamma \in \Gamma\) such that \(a = p \alpha a \beta q \gamma\).

Consider \((\lambda_1 \cap \lambda_2)(a) = \sup_{a=p \alpha q} \min_{q} \{\lambda_1(p), \lambda_2(q)\}\)
\[ \geq \min_{a=p \alpha q} \{\lambda_1(p), \lambda_2(q)\}\]
\[ \geq \min_{a=p \alpha q} \{\lambda_1(a), \lambda_2(a)\}\]
\[ = (\lambda_1 \cap \lambda_2)(a)\]

Hence \((L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)\).

(ii) \(\Rightarrow\) (i) suppose \((L, \lambda_1)\) and \((R, \lambda_2)\) are fuzzy soft \(\Gamma\)-left ideal and fuzzy soft \(\Gamma\)-
right ideal of $S$ such that $(L, \lambda_1) \cap_R (R, \lambda_2) \subseteq (L, \lambda_1) \supseteq (R, \lambda_2)$, by theorem (3.10) $\chi_{F(e)}$ is fuzzy soft $\Gamma$ – left ideal of $S$ and $\chi_{F(e)}$ is fuzzy soft $\Gamma$ – right ideal of $S$. Now by theorem (3.7), we have $\chi_{F(e)\cap_R} = \chi_{F(e)\cap_R}$ and hence

$$1 = (\chi_{F(e)\cap_R})(a)$$

$$= (\chi_{F(e)\cap_R})(a) \cap \chi_{F(e)\cap_R}(a)$$

$$= (\chi_{F(e)\cap_R}) \cap \chi_{F(e)\cap_R}(a)$$

$$= (\chi_{F(e)\cap_R})(a) \text{ by theorem (3.7)}$$

It follows that $a \in (L, \lambda_1) \supseteq (R, \lambda_2) \cap_R (L, \lambda_1) \supseteq (R, \lambda_2)$

The above theorem hence $(F, A)$ is left $\Gamma$ – intra regular.

**Acknowledgement:** The research of the second author is partially supported by UGC-BSR grant: F.25-1/2014-15(BSR)/7 -254/2009(UBSR) dated 20-01-2015 in India.

**References**


