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VARIOUS CARTESIAN PRODUCTS OF VERTEX DEGREE AND EDGE DEGREE IN HESITANCY FUZZY GRAPHS

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Abstract:

In this paper, we introduce Cartesian products for hesitancy fuzzy graphs with a suitable illustration and proper theoretical validation. Also, we establish three new versions of Cartesian products for hesitancy fuzzy graphs.

Key Words: Hesitancy Fuzzy Graphs (HFGs), Cartesian Product of Two Hesitancy Fuzzy Graphs & Degree of Vertex in Hesitancy Fuzzy Graphs

1. Introduction:

Many of the real world problems are very complex and full of unclear information. Rosenfeld [10] developed fuzzy graphs and Mordeson [4] introduced several operations on fuzzy graphs with suitable illustrations to model such complex problems in a graphical manner. Torra [12] introduced a new extension of fuzzy sets called Hesitant Fuzzy Sets to handle the common difficulty that appears in fixing the membership degree of an element from some possible values which leads to a growth of many concepts.

Hesitancy Fuzzy Graphs (HFGs) was introduced by Pathinathan. T et. al in [9]. Also [9] expresses the various introductory concepts of Hesitancy Fuzzy Graphs with suitable illustrations. Then Pathinathan. T and Jon Arockiaraj. J extended Hesitancy Fuzzy Graph such as Constant Hesitancy Fuzzy Graphs [9], Regular Hesitancy Fuzzy Graph (c-HFGs) [7], totally constant and totally regular Hesitant Fuzzy Graphs (tc-HFGs) with some theoretical properties. Concepts like size, order, degree, total degree are also introduced along with theoretical analysis and examples.

Atanassov [2] introduced first five versions of Cartesian product of two Intuitionistic Fuzzy Sets in the year 1999 and Andonov [1] extended it to the sixth version. Then, Varghese et.al introduced five more versions of Cartesian product over two Intuitionistic Fuzzy Sets. Pathinathan. T and Jesintha Rosline [3] introduced vertex degree of Cartesian products for Intuitionistic Fuzzy graph and studied their theoretical properties. This paper introduces various cartesian products for hesitancy fuzzy graph along with the suitable illustration and theoretical description.

The paper is organised as follows. Section Two focuses on the concept of Fuzzy Graph (FG), Hesitancy Fuzzy Graphs (HFGs) and some of the related basic definitions. Section Three provides concepts like Cartesian product of two hesitancy fuzzy graphs with some related theoretical concepts. Section Four deals with the newly introduced Cartesian product types with illustrative examples followed by conclusion in section Five.

2. Basic Definitions and Terminologies:

This section contains some definitions and examples on Hesitancy Fuzzy Graphs and its related topics.

Definition 2.1 (Fuzzy Graph)

Let V be a non empty set. A fuzzy graph is a pair of functions $G(\sigma,\mu)$ where σ is a fuzzy subset of V, μ is a symmetric fuzzy relation on σ.

$$\sigma : V \to [0,1]$$

$$\mu : V \times V \to [0,1]$$

such that $\mu(u,v) \le \sigma(u) \wedge \sigma(v) \, \forall u,v \in V$.

The underlying crisp graph of the fuzzy graph $G(\sigma,\mu)$ is denoted as $G^* : (\sigma^*, \mu^*)$ where σ^* is referred to as the nonempty set V of nodes and $\mu^* = E \subseteq V \times V$.

The crisp graph (V, E) is a special case of the fuzzy graph G with each vertex and edge of (V,E) having degree of membership 1.

Definition 2.2 (Hesitancy Fuzzy Graph)

A Hesitancy Fuzzy Graph is of the form G = (V,E), where

(i) $V = \{v_1, v_2, \ldots v_n\}$ such that $\mu_1 : V \to [0,1], \gamma_1 : V \to [0,1]$ and $\beta_1 : V \to [0,1]$ denote the degree of membership, non-membership and hesitancy of the element $v_i \in V$ respectively and

$$\mu_1(v_i) + \gamma_1(v_i) + \beta_1(v_i) = 1 \; \forall v_i \in V,$$

where $\beta_1(v_i) = 1 - [\mu_1(v_i) + \gamma_1(v_i)]$ and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$ --------(1)

(ii) $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0,1], \gamma_2 : V \times V \to [0,1]$ and $\beta_2 : V \times V \to [0,1]$ are such that,

$$\mu_2(v_i, v_j) \le \min(\mu_1(v_i), \mu_1(v_j)) \quad \text{-------(2)}$$

$$\gamma_2(v_i, v_j) \le \max(\gamma_1(v_i), \gamma_1(v_j)) \quad \text{-------(3)}$$

$$\beta_2(v_i, v_j) \le \min(\beta_1(v_i), \beta_1(v_j)) \quad \text{-------(4)}$$

And $0 \le \mu_2(v_i,v_j) + \gamma_2(v_i,v_j) + \beta_2(v_i,v_j) \le 1 \, \forall (v_i,v_j) \in E$ --------(5)

Notations:

- ✓ $\langle v_i, \mu_{1i}, \gamma_{1i}, \beta_{1i} \rangle$ denotes the vertex, degree of membership, non-membership and hesitancy of the vertex v_i.
- ✓ $\langle e_{ij}, \mu_{2ij}, \gamma_{2ij}, \beta_{2ij} \rangle$ denotes the edge, degree of membership, non-membership and hesitancy of the edge relation $e_{ij} = (v_i, v_j)$ on V.

Example 2.1 (Hesitancy Fuzzy graph)

Consider G = (V, E), where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1\}$, then the hesitancy fuzzy graph is obtained by using the above definition (Definition 2.2).

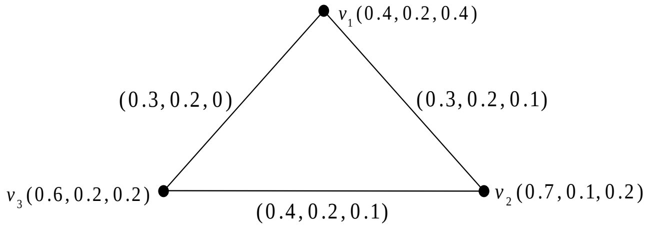

Figure 1: Hesitancy Fuzzy graph

Definition 2.3 (Cartesian product of two fuzzy sets)

Let $\underset{\sim}{A}$ and $\underset{\sim}{B}$ be two non-empty fuzzy sets, the product set or Cartesian product $\underset{\sim}{A} \times \underset{\sim}{B}$ is defined as follows,

$$\underset{\sim}{A} \times \underset{\sim}{B} = \{(a,b) \mid a \in \underset{\sim}{A}, b \in \underset{\sim}{B}\}$$

Definition 2.4 (Cartesian product of two hesitant fuzzy sets)

Let $\underset{\sim}{A} = \langle \mu_{\underset{\sim}{A}}(x), \gamma_{\underset{\sim}{A}}(x), \beta_{\underset{\sim}{A}}(x) \rangle$ and $\underset{\sim}{B} = \langle \mu'_{\underset{\sim}{A}}(x), \gamma'_{\underset{\sim}{A}}(x), \beta'_{\underset{\sim}{A}}(x) \rangle$ be two non-empty hesitancy fuzzy sets, where $\mu_1, \mu'_1 : V \to [0,1]$, $\gamma_1, \gamma'_1 : V \to [0,1]$ and $\beta_1, \beta'_1 : V \to [0,1]$ denotes the membership, non-membership and hesitant values, satisfies the conditions of the above definition 2.2. Then the cartesian product $\otimes$ is defined by,

$$\underset{\sim}{A} \otimes \underset{\sim}{B} = \{(\mu_{\underset{\sim}{A}}(x).\mu'_{\underset{\sim}{A}}(x)), (\gamma_{\underset{\sim}{A}}(x).\gamma'_{\underset{\sim}{A}}(x)), (\beta_{\underset{\sim}{A}}(x).\beta'_{\underset{\sim}{A}}(x))\}$$

and the other cartesian products is described as follows (Table 2.1);

Table 2.1: Notations of Cartesian product

S.No	$\otimes$	Description
1	$\cdot_m$	Multiplication
2	$\oplus_m$	Algebraic Product for membership values
3	$\oplus_{nm}$	Algebraic Product for non-membership values
4	$\oplus_h$	Algebraic Product for hesitant values
5	$\cap_m$	Intersection
6	$\cup_m$	union
7	AM	Arithmetic Mean
8	GM	Geometric Mean
9	HM	Harmonic mean
10	QM	Quadratic Mean
11	CM	Cubic Mean
12	h	Heronian Mean
13	$\inf m$	Infimum
14	$\sup m$	Supremum

3. Cartesian Product of Two Hesitancy Fuzzy Graphs:

Definition 3.1 (Cartesian product of two hesitancy fuzzy graphs)

The Cartesian product of two hesitancy fuzzy graphs G_1 and G_2 is defined as a HFG,

$$G = G_1 \times G_2 : (V, E)$$

where, $V = V_1 \times V_2$ and $E = \{(u_1, u_2)(v_1, v_2) / u_1 = v_1 \,\&\, u_2v_2 \in E_2 \text{ or } u_2 = v_2 \,\&\, u_1v_1 \in E_1\}$ with;

$$\langle(\mu_1\times\mu_1'),(\gamma_1\times\gamma_1'),(\beta_1\times\beta_1')\rangle(u_1,u_2)=\langle(\mu_1(u_1)\wedge\mu_1'(u_2)),(\gamma_1(u_1)\vee\gamma_1'(u_2)),(\beta_1(u_1)\wedge\beta_1'(u_2))\rangle$$

$\forall$ every $(u_1,u_2)\in V$.

and;

$$\langle(\mu_2\times\mu_2'),(\gamma_2\times\gamma_2'),(\beta_2\times\beta_2')\rangle(u_1,u_2)(v_1,v_2)=$$

$$\begin{cases}\langle(\mu_1(u_1)\wedge\mu_2'(u_2,v_2)),(\gamma_1(u_1)\vee\gamma_2'(u_2,v_2)),(\beta_1(u_1)\wedge\beta_2'(u_2,v_2))\rangle & ;\text{if } u_1=v_1 \,\&\, (u_2,v_2)\in E_2\\ \langle(\mu_1'(u_2)\wedge\mu_2(u_1,v_1)),(\gamma_1'(u_2)\vee\gamma_2(u_1,v_1)),(\beta_1'(u_2)\wedge\beta_2(u_1,v_1))\rangle & ;\text{if } u_2=v_2 \,\&\, (u_1,v_1)\in E_2\\ \langle 0,0\rangle & ;\text{Otherwise}\end{cases}$$

Example 3.1 (Cartesian Product of Two Hesitancy Fuzzy Graphs)

Let G_1 and G_2 be two hesitancy graphs with $\{u_1,v_1\}$ and $\{u_2,v_2\}$ be the vertex sets respectively. Then the Cartesian product between G_1 and G_2 is given by $G_1\times G_2$ as follows;

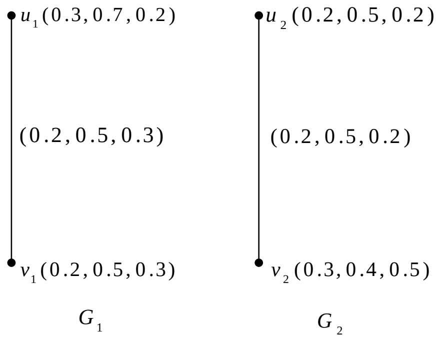

Figure 2: Cartesian product of two hesitancy fuzzy graphs G_1 and G_2

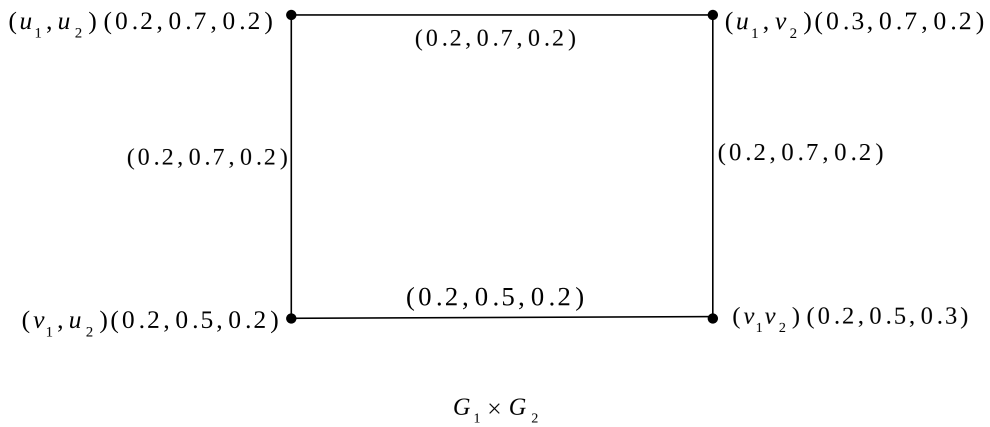

Figure 3: Cartesian product $G_1\times G_2$ of two hesitancy fuzzy graphs

Theorem 3.1

Let $G_1 : \langle(v_i,\mu_1,\gamma_1,\beta_1),(e_{ij},\mu_2,\gamma_2,\beta_2)\rangle$ and $G_2 : \langle(v_i,\mu_1',\gamma_1',\beta_1'),(e_{ij},\mu_2',\gamma_2',\beta_2')\rangle$ be two hesitancy fuzzy graphs.

If $\mu_1\geq\mu_2',\gamma_1\leq\gamma_2',\beta_1\leq\beta_2'$ and $\mu_1'\geq\mu_2,\gamma_1'\leq\gamma_2,\beta_1'\geq\beta_2$ then $d_{G_1XG_2}(u_1,u_2)=d_{G_1}(u_1)+d_{G_2}(u_2)$.

Proof:

$$d_{G_1XG_2}(u_1,u_2)=\langle d_{\mu_2X\mu_2'}(u_1,u_2),d_{\gamma_2X\gamma_2'}(u_1,u_2),d_{\beta_2X\beta_2'}(u_1,u_2)\rangle$$

$$=\left\langle \sum_{u_1=v_1,(u_2,v_2)\in E_2}\mu_1(u_1)\wedge\mu_2'(u_2,v_2),\sum_{u_1=v_1,(u_2,v_2)\in E_2}\gamma_1(u_1)\vee\gamma_2'(u_2,v_2),\sum_{u_1=v_1,(u_2,v_2)\in E_2}\beta_1(u_1)\wedge\beta_2'(u_2,v_2)\right\rangle+$$

$$\left\langle \sum_{u_2=v_2,(u_1,v_1)\in E_2}\mu_1'(u_2)\wedge\mu_2(u_1,v_1),\sum_{u_2=v_2,(u_1,v_1)\in E_2}\gamma_1'(u_2)\vee\gamma_2(u_1,v_1),\sum_{u_2=v_2,(u_1,v_1)\in E_2}\beta_1'(u_2)\wedge\beta_2(u_1,v_1)\right\rangle$$

$$=\left\langle \sum_{u_1=v_1}\mu_2'(u_2,v_2),\sum_{u_1=v_1}\gamma_2'(u_2,v_2),\sum_{u_1=v_1}\beta_2'(u_2,v_2)\right\rangle+\left\langle \sum_{u_2=v_2}\mu_2(u_1,v_1),\sum_{u_2=v_2}\gamma_2(u_1,v_1),\sum_{u_2=v_2}\beta_2(u_1,v_1)\right\rangle$$

$$=d_{G_1}(u_1)+d_{G_2}(u_2).$$

Theorem 3.2

If $G=\langle(v_i,\mu_1,\gamma_1,\beta_1),(e_{ij},\mu_2,\gamma_2,\beta_2)\rangle$ and $G_2:\langle(v_i,\mu_1',\gamma_1',\beta_1'),(e_{ij},\mu_2',\gamma_2',\beta_2')\rangle$ are two hesitancy fuzzy graphs, such that

$$\mu_1\le\mu_2',\gamma_1\ge\gamma_2' \text{ and } \beta_1\le\beta_2', \text{then } \mu_1'\ge\mu_2,\gamma_1'\le\gamma_2 \text{ and } \beta_1'\ge\beta_2.$$

Proof:

By the definition of hesitancy fuzzy graphs;

$$\mu_2(v_i,v_j)\le\min(\mu_1(v_i),\mu_1(v_j))----------(1)$$

$$\gamma_2(v_i,v_j)\le\max(\gamma_1(v_i),\gamma_1(v_j))----------(2)$$

$$\beta_2(v_i,v_j)\le\min(\beta_1(v_i),\beta_1(v_j))----------(3)$$

Therefore;

From (1), we have

$$\mu_2\le\max\mu_1 \text{ and } \min\mu_2\le\mu_1$$

From (2), we have

$$\gamma_2\ge\min\gamma_1 \text{ and } \max\gamma_2\ge\gamma_1$$

From (3), we have

$$\beta_2\le\max\beta_1 \text{ and } \min\beta_2\le\beta_1$$

Also we have;

$$\mu_1\le\mu_2';\max\mu_1\le\min\mu_2'$$

$$\gamma_1\ge\gamma_2';\min\gamma_1\ge\max\gamma_2'$$

$$\beta_1\le\beta_2';\max\beta_1\le\min\beta_2'$$

Hence

$$\mu_2\le\max\mu_1\le\min\mu_2'\le\mu_1'$$

$$\gamma_2\ge\min\gamma_1\ge\max\gamma_2'\ge\gamma_1'$$

$$\beta_2\le\max\beta_1\le\min\beta_2'\le\beta_1'$$

Thus, $\mu_1'\ge\mu_2,\gamma_1'\le\gamma_2$ and $\beta_1'\ge\beta_2$.

4. Other Cartesian Products:

Definition 4.1

Let X be an universal set and let V be an HFS over X in the form,
$V=\{(v_i,\mu_i(v_i),\gamma_i(v_i),\beta_i(v_i))\,|\,v_i\in V\}$ such that $0\le\mu_i(v_i)+\gamma_i(v_i)+\beta_i(v_i)\le1$. The following are the different types of Cartesian products (in crisp sense) of n subsets $V_1,V_2,V_3,\dots V_n$ of V over X :

$$v_i \times_1 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \mu_i.\mu_j, \gamma_i.\gamma_j, \beta_i.\beta_j \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_2 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \mu_i + \mu_j - \mu_i.\mu_j, \gamma_i.\gamma_j, \beta_i.\beta_j \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_3 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \mu_i.\mu_j, \gamma_i + \gamma_j - \gamma_i.\gamma_j, \beta_i.\beta_j \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_4 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \mu_i.\mu_j, \gamma_i.\gamma_j, \beta_i + \beta_j - \beta_i.\beta_j \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_5 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \min(\mu_i, \mu_j), \max(\gamma_i, \gamma_j), \min(\beta_i, \beta_j) \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_6 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \max(\mu_i, \mu_j), \min(\gamma_i, \gamma_j), \min(\beta_i, \beta_j) \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_7 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \frac{\mu_i + \mu_j}{2}, \frac{\gamma_i + \gamma_j}{2}, \frac{\beta_i + \beta_j}{2} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_8 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \sqrt{\mu_i \mu_j}, \sqrt{\gamma_i \gamma_j}, \sqrt{\beta_i \beta_j} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_9 v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \frac{2\mu_i \mu_j}{\mu_i + \mu_j}, \frac{2\gamma_i \gamma_j}{\gamma_i + \gamma_j}, \frac{2\beta_i \beta_j}{\beta_i + \beta_j} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_{10} v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \sqrt{\frac{\mu_i^2 + \mu_j^2}{2}}, \sqrt{\frac{\gamma_i^2 + \gamma_j^2}{2}}, \sqrt{\frac{\beta_i^2 + \beta_j^2}{2}} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_{11} v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \sqrt[3]{\frac{\mu_i^3 + \mu_j^3}{2}}, \sqrt[3]{\frac{\gamma_i^3 + \gamma_j^3}{2}}, \sqrt[3]{\frac{\beta_i^3 + \beta_j^3}{2}} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_{12} v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \frac{\mu_i + \mu_j + \sqrt{\mu_i \mu_j}}{3}, \frac{\gamma_i + \gamma_j + \sqrt{\gamma_i \gamma_j}}{3}, \frac{\beta_i + \beta_j + \sqrt{\beta_i \beta_j}}{3} \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_{13} v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \min(1, \mu_i + \mu_j), \max(0, \gamma_i + \gamma_j - 1), \min(0, \beta_i + \beta_i - 1) \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

$$v_i \times_{14} v_j = \left\{ \left\langle \langle v_i, v_j \rangle, \max(0, \mu_i + \mu_j - 1), \min(1, \gamma_i + \gamma_j), \min(0, \beta_i + \beta_i - 1) \right\rangle | \langle v_i, v_j \rangle \in V \times V \right\}$$

It must be noted that $v_i \times_t v_j$ is an IFS, where t = 1,2,3,4,5,6,7,8,9,10,11,12,13,14.

Also we have,

$E \subseteq V \times V$ where $\mu_{ij} : V \to [0,1]$, $\gamma_{ij} : V \to [0,1]$ and $\beta_{ij} : V \to [0,1]$ are such that

$$\mu_{ij} \le \mu_i \otimes \mu_j ; \gamma_{ij} \le \gamma_i \otimes \gamma_j ; \beta_{ij} \le \beta_i \otimes \beta_j$$

where μ_{ij}, γ_{ij} and β_{ij} are the membership, non- membership and hesitancy values of the edge (v_i, v_j) such that $0 \le \mu_{ij} + \gamma_{ij} + \beta_{ij} \le 1$

By using the notations defined in the definition 2.4, the below table represents various Cartesian products over hesitancy fuzzy graphs.

Table 4.1: Various Cartesian Products over hesitancy fuzzy graphs

S.No	$\otimes$	$\mu_i \otimes \mu_j$	$\gamma_i \otimes \gamma$	$\beta_i \otimes \beta_j$
1	$\cdot_m$	$\mu_i.\mu_j$	$\gamma_i.\gamma_j$	$\beta_i.\beta_j$

2	$\oplus_m$	$\mu_i + \mu_j - \mu_i.\mu_j$	$\gamma_i.\gamma_j$	$\beta_i.\beta_j$
3	$\oplus_{nm}$	$\mu_i.\mu_j$	$\gamma_i + \gamma_j - \gamma_i.\gamma_j$	$\beta_i.\beta_j$
4	$\oplus_h$	$\mu_i.\mu_j$	$\gamma_i.\gamma_j$	$\beta_i + \beta_j - \beta_i.\beta_j$
5	$\cap_m$	$\min(\mu_i, \mu_j)$	$\max(\gamma_i, \gamma_j)$	$\min(\beta_i, \beta_j)$
6	$\cup_m$	$\max(\mu_i, \mu_j)$	$\min(\gamma_i, \gamma_j)$	$\min(\beta_i, \beta_j)$
7	AM	$\frac{\mu_i + \mu_j}{2}$	$\frac{\gamma_i + \gamma_j}{2}$	$\frac{\beta_i + \beta_j}{2}$
8	GM	$\sqrt{\mu_i \mu_j}$	$\sqrt{\gamma_i \gamma_j}$	$\sqrt{\beta_i \beta_j}$
9	HM	$\frac{2\mu_i \mu_j}{\mu_i + \mu_j}$	$\frac{2\gamma_i \gamma_j}{\gamma_i + \gamma_j}$	$\frac{2\beta_i \beta_j}{\beta_i + \beta_j}$
10	QM	$\sqrt{\frac{\mu_i^2 + \mu_j^2}{2}}$	$\sqrt{\frac{\gamma_i^2 + \gamma_j^2}{2}}$	$\sqrt{\frac{\beta_i^2 + \beta_j^2}{2}}$
11	CM	$\sqrt[3]{\frac{\mu_i^3 + \mu_j^3}{2}}$	$\sqrt[3]{\frac{\gamma_i^3 + \gamma_j^3}{2}}$	$\sqrt[3]{\frac{\beta_i^3 + \beta_j^3}{2}}$
12	h	$\frac{\mu_i + \mu_j + \sqrt{\mu_i \mu_j}}{3}$	$\frac{\gamma_i + \gamma_j + \sqrt{\gamma_i \gamma_j}}{3}$	$\frac{\beta_i + \beta_j + \sqrt{\beta_i \beta_j}}{3}$
13	$\inf m$	$\min(1, \mu_i + \mu_j)$	$\max(0, \gamma_i + \gamma_j - 1)$	$\min(0, \beta_i + \beta_i - 1)$
14	$\sup m$	$\max(0, \mu_i + \mu_j - 1)$	$\min(1, \gamma_i + \gamma_j)$	$\min(0, \beta_i + \beta_i - 1)$

Note: If $\mu_{ij} = \gamma_{ij} = 0$, for some i and j, then there is no edge between v_i and v_j, and it is indexed by $\langle 0,1 \rangle$. Otherwise there exists edge between v_i and v_j.

Illustration 4.1

For the above definition 4.1, we have the following illustration. Consider G = (V,E), where $V = \{v_1, v_2, v_3\}$ and $E = \{v_1v_2, v_2v_3, v_3v_1\}$, then the various Cartesian product is follows as;

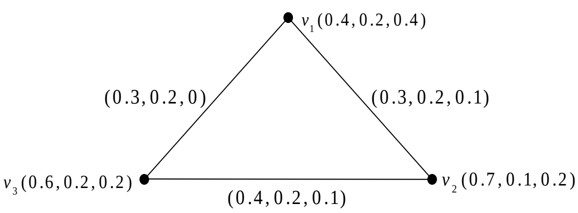

Figure 4: Hesitancy Fuzzy Graph

Table 4.2: Cartesian products for Figure 4

S.No	$\otimes$	$\mu_i \otimes \mu_j$	$\gamma_i \otimes \gamma$	$\beta_i \otimes \beta_j$
1	$\cdot_m$	0.28	0.02	0.08
2	$\oplus_m$	0.82	0.02	0.08
3	$\oplus_{nm}$	0.28	0.28	0.08
4	$\oplus_h$	0.28	0.02	0.52
5	$\cap_m$	0.4	0.2	0.2
6	$\cup_m$	0.7	0.1	0.2

7	AM	0.55	0.15	0.3
8	GM	0.53	0.14	0.28
9	HM	0.51	0.13	0.27
10	QM	0.57	0.16	0.32
11	CM	0.52	0.31	0.29
12	h	0.54	0.15	0.29
13	$\inf m$	1	0.0	0.4
14	$\sup m$	0.1	0.3	0.4

5. Conclusion:

In this paper we have defined a Cartesian product for Hesitancy Fuzzy Graph with suitable illustrations. Also few theoretical concepts are derived. Further, we established various Cartesian products over hesitancy fuzzy graph with proper illustrations. In the upcoming article on this hesitancy fuzzy graph, we will develop the other valid theoretical concepts with relevant examples.

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