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THE SYSTEM OF DIFFERENTIAL EQUATIONS IN PREY PREDATOR MODEL

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Abstract:

Predation is the interaction involved in this paper. Prediction is the species interaction when one species, the predator, eats another species, the prey, as a source of food. Predation relationship exists in ecological niches throughout the world. This is the species interaction which will be mathematically analyzed and embodied in this work. The importance of studying predation lies in the direct possible effects of predation relationships, especially co evolution, natural selection, and even the possibility of extinction.

Key Words: The Lotka Volterra Equations, Prey & Predator

Prey Predator Model:

The Lotka Volterra Model:

The Lotka volterra equations ,also known as the predator –prey equations ,are a pair of first-order, non –linear, differential equations frequency used to describe the dynamics of biological systems in which two species interact ,one as a predator and the other as prey.

The populations change through time according to the pair of equations:

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -CY + Dxy$$

- x is the number of prey (for example :rabbits)
- y is the number of some predators (for example, foxes)
- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ represent the growth rates of the two populations over time,
- t represent time, and
- A,B,C,D are positive real parameters describing the interaction of the two species:

  A - Growth rate of prey
  B - Searching efficiency or attack rate
  C - Predator mortality rate
  D - Growth rate of predator or predator's ability at turning food into offspring

Mathematical Formulation:

We being by looking at what happens to the predator population in the absence of prey, without food resources, their numbers are expected to decline exponentially, as described by the following equations.

$$\frac{dy}{dt} = -Cy$$

This equations uses the product of the number of predators(y)and the predator mortality rate (-C) to describe the rate of decrease(because of the minus sign on the right hand side of the equation) of the predator population(y) with respect to time (t).In the presence of prey, however, this decline is opposed by the predator birth rate, Dxy,

which is determined by the product of the number of predators [y] times the number of prey [x] and by the predator's ability to turn food into offspring (D). As predator and prey numbers (y and x, respectively) increase, their encounters become more frequent.
The equation describing the predator population dynamics becomes

$$\frac{dy}{dt} = -Cy + Dxy$$

The product Dy is the predator's numerical response, or the per capita increase as a function of prey abundance. The entire term, Dxy, tells us that increase in the predator population are proportional to the product of predator and prey abundance. Turning to the prey population, we would expect that without predation, the number of prey would increase exponentially. The following equation describes the rate of increase of the prey population with respect to time, where A is the growth rate of the prey population, and x is the abundance of the prey population

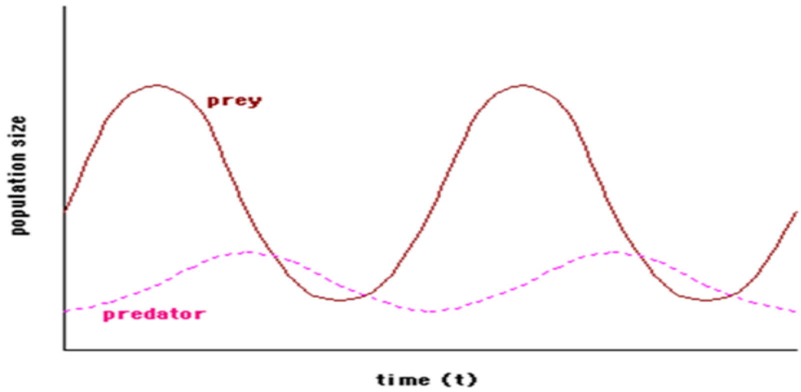
$$\frac{dx}{dt} = \mathrm{Ax}$$

In the presence of predators, however, the prey population is prevented from increasing exponentially. The term for consumption rate (Bxy) which is the attack rate [B] multiplied by the number of predators [y] times the number of prey [x], describes prey mortality, and the population dynamics of the prey can be described by the equation

$$\frac{dx}{dt} = Ax - Bxy$$

The product of B and y is the predator's functional response, or rate of prey capture as a function of prey abundance. Here the term Bxy reflects the fact that losses from the prey population due to predation are proportional to the product of predator and prey abundances.

The equations describe predator and prey population dynamics in the presence of one another, and together make up the Lotka Volterra predator – prey model. The model predicts a cyclical relationship between predator and prey numbers as the number of predators (y) increase so does the consumption rate (Bxy),tending to reinforce the increase in y `Increase in consumption rate, however, has an obvious consequence - a decrease in the number of prey (x), which in turn causes y (and therefore Bxy) to decrease. As Bxy decrease the prey population is able to recover, and x increases. Now y can increase, and the cycle begins again. This graph shows the cyclical relationships predicted by the model for hypothetical predator and prey populations.



Hypothetical Predator and Prey Populations

Talking the Lotka Volterra Equations:

Now that we have investigated a slightly more straight forward situation, we will work directly with the Lotka volterra equations. As we started earlier, there is no guarantee that a conserved quantity exists for a system, but assuming one does exist ,we will begin by finding the conserved quantity for the system,

$$\frac{dx}{dt} = Ax - Bxy$$

$$\frac{dy}{dt} = -Cy + Dxy$$

to start, multiply both sides of each equation by 1/xy to get,

$$0 = \frac{dx}{dt}\frac{1}{xy} = \frac{A}{Y} - B$$

$$\frac{dy}{dt}\frac{1}{xy} = -\frac{C}{x} + D$$

Multiplying both sides of each equation by the appropriate derivative, we get

$$\frac{1}{xy}\frac{dx}{dt}\frac{dy}{dt} = \frac{dy}{dt}\left(\frac{A}{y} - B\right)$$

$$\frac{1}{xy}\frac{dx}{dt}\frac{dy}{dt} = \frac{dx}{dt}\left(-\frac{C}{x} + D\right)$$

Subtracting the bottom equation from the top yields

$$0 = \frac{dy}{dt}\left(\frac{A}{y} - B\right) + \frac{dx}{dt}\left(\frac{C}{x} - D\right)$$

Like before, $\frac{dE}{dt} = 0$, so the right hand side must match our chain-rule expansion for dE/dt,

$$\frac{d}{dt}E(x,y) = \frac{dy}{dt}\frac{\partial}{\partial y}E(x,y)\frac{dx}{dt}\frac{\partial}{\partial x}E(x,y)$$

These two equations tell us that if,

$$\frac{\partial}{\partial y}E(x,y) = \frac{A}{y} - B$$

$$\frac{\partial}{\partial x}E(x,y) = \frac{C}{x} - D$$

like before, E (x(t),y(t)) is a constant as t varies . Thus we can write that

$$E(x,y) = \int\left(\frac{A}{y} - B\right)\partial y = \int\left(\frac{C}{x} - D\right)\partial x$$

Integrating we get,

E(x,y) = Alny – By + f(x) = Clnx – Dx + g(y)

Again we see arbitrary functions of integration, which are eliminated when we take a partial derivative just like how ordinary integration introduces arbitrary constants, which ordinary derivatives wipe out. Merging the sides of the above equation as before, we get that

E(x,y) = Alny + Clny – By – Dx + M

with a little algebraic manipulation ,like before ,we see that

L – M = Alny + Clnx – By – Dx and so

$$E(x,y) = A\ln y + C\ln x - By - Dx = K$$

where K = L – M is a constant that depends on initial conditions and not on t.

Like we stated with the simpler system, this is not the only possible way to define a conserved quantity for the Lotka – Volterra equations. For example, it is often the case that $e^{E(x,y)}$(with E(x,y) being the above version) is the chosen definition for the Lotka – Volterra equations conserved quantity ,which is perfectly valid since it will still always be constant as t varies.

However, since the way we have defined E(x,y) is also perfectly valid, and since it is a common version, we will stick to it for the remainder of our analysis.

Instead of straightforward polynomial functions ,we see natural logarithms, although there are linear terms as before .The reason for this difference stems from the differences in the initial equations : the simpler system had "-y" and "-x" where the Lotka – Volterra equation have " A/y" and " C/x"(compare the respective integrals in the derivations) . One thing to take from the conserved quantity for the Lotka Volterra equations is that, because of the natural logarithms, it is not defined anywhere where $x \le 0$ or $y \le 0$; hence, for any value of K,x and y will never go below 0 if the system starts out with E(x,y) = K (this matches our intuitive notion that there cannot be negative numbers of either predator or prey)

More Prey Predator Models:

Volterra's principle has spectacular applications to insecticide treatments, which destroy both insect predators and their insect prey. It implies that the application of insecticides will actually increase the population of those insects which are kept in control by other predatory insects. A remarkable confirmations comes from the cottony cushion scale insect (Icery purchase), when accidentally introduced from Australia in 1868, threatenedto destroy the American citrus industry. Thereupon, its natural Australian predator, a ladybird beetle (Novius Cardinalis) was introduced, and the beetles reduced the scale insects to a low level. When DDT was discovered to kill scale insects, it was applied by the orchardists in the hope of further reducing the scale insects. However, in agreement with Volterra's principal, the effect was an increase of the scale insect.

Oddly enough many ecologists and biologists refused to accept Volterra's model as accurate. They pointed to the fact that the oscillatory behavior predicted by Volterra's model is not observed in most predator- prey system. Rather, more predator-prey systems tend to equilibrium states as time evolves. Our answer to these critics is that the system of differential equations is not intended as a model of the general predator – prey interaction. This is because the prey and predators do not complete intensively among themselves for their available resources.

A more general model of predator – prey interactions is the system of differential equations,

$$\frac{dx}{dt} = Ax - Bxy - Ex^2; \frac{dy}{dt} = -Cy + Fy^2 \qquad (*)$$

Here the term Ex reflects the internal competition of the prey x for their limited external resources, and the term Fy reflects the competition among the predators for the limited number of prey. However, some ecologists and biologist even refuse to accept the more general model (*) as accurate. As a counter example, they cite the experiments of the mathematical biologist G.F.Gause. In these experiments, the populations was composed of two species of protozoa, one of which, Didymium nasatum, feeds on the other,paramecium caudatum .In all of Gause's experiments the

Didymium quickly destroyed the paramecium and then died of starvation. This situation cannot be modeled by the system of equations (*) , since no solution of (*) with x(0)y(0) ≠ 0 can reach x = 0 or y = 0 in finite term.

Our answer to these critics is that the Didymium is a special, and a typical type of predator. On the one hand, they are ferocious attackers and require a tremendous amount of food a Didymium demands fresh paramecium every three hours. On the other hand, the Didymium does not perish from an insufficient supply of paramecium every three hours. On the other hand, the Didymium does not perish from an insufficient supply of paramecium. They continue to multiply, but give birth to smaller offspring. Thus the system of equations (*) does not accurately model the interaction of paramecium and Didymium. A better model, in this case, is the system of differential equations,

$$\frac{dx}{dt} = Ax - B\sqrt{xy}; \frac{dy}{dt} = \begin{cases} D\sqrt{xy}, x \neq 0 \\ -Cy, x = 0 \end{cases} \qquad (**)$$

Finally, we mention that there are several predator – prey interactions in nature with cannot be modeled by any system of ordinary differential equations. These situations occur when the prey are provided with a refuge that the inaccessible to the predators.

In these situations, it is impossible to make any definitive statements about the future number of predators and prey, since we cannot predict how many prey will leave their refuge.

In other words, this process is now random, rather than deterministic, and therefore cannot be modeled by a system of ordinary differential equations. This was verified directly in a famous experiment of gauze .He placed five paramecium and three Didymium in each of thirty identical test tubes, and provided the paramecium with a refuge from the Didymium. Two days, later , he found the predators dead in four tubes ,and a mixed population containing from two to thirty-eight paramecium in the remaining twenty-six tubes.

Example:

The following set of ordinary differential equations describes the dynamics of a predator-prey system.

$$\frac{dx}{dt} Ax - Bxy \quad ;x(0) = 3 \text{ Rabbits (prey)}$$

$$\frac{dy}{dt} = Dxy - Cy \quad ; y(0) = 2 \text{ Foxes (predator)}$$

where, A = 2,B=2,C=1,D=1

We find x(0.1) and y(0.1) by manually advancing these ODEs from the initial position (at t= 0). To the next step (at t = 0) with Euler`s method.

Solution:

Euler`s method for numerical integration of ODEs:

With A = 2,B = 2, C= 1, D = 1

The dynamic equations are:

$$\frac{dx}{dt} = 2x - 2xy$$

$$\frac{dy}{dt} = xy - y$$

start with x =3 and y =2, and the slopes at this starting point are :

$\frac{dx}{dt}$ = 2(3) – 2(3)(2) = 6 – 12 = -6

$\frac{dy}{dt}$ = 3(2) – 2 = 6 -2 = 4

with, h = 0.1

x(at t = 0.1) = x (0) + h * $\left(\frac{dx}{dt}\right)$ = 3 + 0.1 * (-6) = 2.4

y(at t = 0.1) = y (0) + h * $\left(\frac{dy}{dt}\right)$ = 2 + 0.1 * 4 = 2.4

This the solution of this problem but some times which is not accurate

Conclusion:

First of all one of the main conclusions of this work is that in the realm of biological mathematics, it is possible to mathematically represent the population variations of predation relationship to a certain extent of accuracy. This can be done using **Lotka Volterra** model. This system of linear first order differential equations can be used to done interpret analytically and graphically the cyclic fluctuations of the species populations. There is a level of error in the theoretical versus actual data , but the overall form, structure, and time of the fluctuations is constant, and errors are most probably due to external variables unaccounted for in the model.This model has simply broken the ground of endless possibility in the biological mathematics world. It is the backbone of the science, proven by tests and time, and the future of this science has incredible potential.

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