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ON THE NEGATIVE PELL EQUATION

$$y^2 = 33x^2 - 8$$

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Abstract:

The binary quadratic equation represented by the negative pellian $y^2 = 33x^2 - 8$ *is analyzed for its distinct integer solutions. A few interesting relations among the solutions are also given. Further, employing the solutions of the above hyperbola, we have obtained solutions of other choices of hyperbolas, parabolas and special Pythagorean triangle.*

Key Words: Binary Quadratic, Hyperbola, Parabola, Integral Solutions & Pell Equation.

1. Introduction:

Diophantine equation of the form $y^2 = Dx^2 + 1$, where D is a given positive square-free integer is known as pell equation and is one of the oldest Diophantine equation that has interesting mathematicians all over the world, since antiquity, J. L. lagrange proved that the positive Pell equation $y^2 = Dx^2 + 1$ has infinitely many distinct integer solutions where as the negative pell equation $y^2 = Dx^2 - 1$ does not always have a solution. In [1], an elementary proof of a ceriterium for the solvability of the pell equation $x^2 - Dy^2 = -1$ where D is any positive non-square integer has been presented. For examples the equations $y^2 = 3x^2 - 1, y^2 = 7x^2 - 4$ have no integer solutions, where as $y^2 = 65x^2 - 1, y^2 = 202x^2 - 1$ have integer solutions. In this context, one may refer [2- 12].More specifically, one may refer " The On-line Encyclopedia of integer sequences" (A031396, A130226, A031398) for values of D for which the negative pell equation $y^2 = Dx^2 - 1$ is solvable or not.

In this communication, the negative Pell equation given by $y^2 = 33x^2 - 8$ is considered and infinitely many integer solutions are obtained. A few interesting relations among the solutions are presented.

2. Method of Analysis:

The negative Pell equation representing hyperbola under consideration is

$$y^2 = 33x^2 - 8 \tag{1}$$

Whose smallest positive integer solution is

$$x_0 = 1, y_0 = 5$$

To obtain the other solutions of (1), consider the Pell equation

$$y^2 = 33x^2 + 1$$

Whose general solution is given by

$$\tilde{x}_n = \frac{1}{2\sqrt{33}} g_n, \ \tilde{y}_n = \frac{1}{2} f_n$$

where

$$f_n = (23+4\sqrt{33})^{n+1} + (23-4\sqrt{33})^{n+1}$$
$$g_n = (23+4\sqrt{33})^{n+1} - (23-4\sqrt{33})^{n+1}$$

Applying Brahmagupta lemma between (x_0,y_0) and $(\tilde{x}_n, \tilde{y}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = \frac{1}{2} f_n + \frac{5}{2\sqrt{33}} g_n$$
$$y_{n+1} = \frac{5}{2} f_n + \frac{\sqrt{33}}{2} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 46x_{n+2} + x_{n+3} = 0$$
$$y_{n+1} - 46y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x and y satisfying (1) are given in the following table below:

n	x_n	y_n
-1	1	5
0	43	247
1	1977	11357
2	90899	522175
3	4463197	25881905

From the above table, we observe some interesting relations among the solutions which are presented below:

1. x_n and y_n are always odd.
2. Each of the following expressions is a nasty number
   - $\frac{3}{2}(33x_{2n+2} - 5y_{2n+2} + 8)$
   - $\frac{3}{736}(22714x_{2n+2} - 10x_{2n+4} + 2944)$
   - $\frac{3}{16}(494x_{2n+2} - 10x_{2n+3} + 64)$
   - $\frac{3}{4}(66x_{2n+2} - 10y_{2n+2} + 16)$
   - $\frac{3}{92}(2838x_{2n+2} - 10y_{2n+3} + 368)$
   - $\frac{3}{4228}(130482x_{2n+2} - 10y_{2n+4} + 16912)$
   - $\frac{3}{16}(22714x_{2n+3} - 494x_{2n+4} + 64)$
   - $\frac{3}{92}(66x_{2n+3} - 494y_{2n+2} + 368)$

- $\frac{3}{4}(2838x_{2n+3} - 494y_{2n+3} + 16)$

3. $\frac{1}{4}(33x_{3n+3} - 5y_{3n+3}) + 3\left[\left(23 + 4\sqrt{33}\right)^{n+1} + \left(23 - 4\sqrt{33}\right)^{n+1}\right]$ is a cubical integer.
4. $x_{n+3} - 184y_{n+1} = 1057x_{n+1}$
5. $184y_{n+1} - 8y_{n+2} = -1056x_{n+1}$
6. $184y_{n+2} - 8y_{n+3} = -1056x_{n+2}$
7. $46x_{n+3} - 8y_{n+3} = 2x_{n+2}$
8. $x_{n+3} - 8y_{n+2} = x_{n+1}$
9. $8y_{n+1} - 8y_{n+3} = -2112x_{n+2}$
10. $46x_{n+3}y_{n+1} - 2114x_{n+2}y_{n+1} = 8y_{n+1}{}^2$
11. $x_{n+1}x_{n+3} - 184x_{n+1}y_{n+1} = 1057x_{n+1}{}^2$
12. $46x_{n+3}x_{n+2} - 8y_{n+1}x_{n+2} = 2114x_{n+2}{}^2$
13. $184y_{n+2}x_{n+1} - 8y_{n+3}x_{n+1} = -1056x_{n+1}x_{n+2}$
14. $184y_{n+1}y_{n+2} - 8y_{n+1}y_{n+3} = -1056y_{n+1}x_{n+2}$
15. $368y_{n+3} - 2112x_{n+1} = 16912y_{n+2}$
16. $16y_{n+3} - 2112x_{n+2} = 368y_{n+2}$
17. $368y_{n+3} - 2112x_{n+3} = 16y_{n+2}$
18. $48576x_{n+1} - 2112x_{n+2} = -8448y_{n+1}$
19. $2112x_{n+1} - 2112x_{n+3} = -16896y_{n+2}$
20. $48576x_{n+2} - 2112x_{n+3} = -8448y_{n+2}$
21. $368y_{n+1}y_{n+3} - 2112x_{n+1}y_{n+1} = 16912y_{n+1}y_{n+2}$
22. $16y_{n+1}y_{+3} - 97152y_{n+1}x_{n+1} = 16912y_{n+1}^2$
23. $48576y_{n+1}x_{n+2} - 2112x_{n+3}y_{n+1} = -8448y_{n+1}y_{n+2}$
24. $48576x_{n+2}y_{n+2} - 2112x_{n+3}y_{n+2} = -8448y_{n+2}^2$
25. $16y_{n+2}y_{n+3} - 2112x_{n+2}y_{n+2} = 368y_{n+2}^2$

3. Remarkable Observations:

1. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below:

S.No	Hyperbola	x, y
1.	$x^2 - y^2 = 8667136$	$\frac{1}{1472}(22714x_{n+1} - 10x_{n+3})$, $\frac{1}{1472\sqrt{33}}(66x_{n+3} - 130482x_{n+1})$
2.	$x^2 - y^2 = 64$	$\frac{1}{16}(33x_{n+1} - 5y_{n+1})$, $\frac{33}{16}(y_{n+1} - 5x_{n+1})$

3.	$x^2 - y^2 = 4096$	$\frac{1}{32}(494x_{n+1} - 10x_{n+2})$, $\frac{1}{32\sqrt{33}}(66x_{n+2} - 2838x_{n+1})$
4.	$x^2 - y^2 = 256$	$\frac{1}{8}(66x_{n+1} - 10y_{n+1})$, $\frac{1}{8\sqrt{33}}(66y_{n+1} - 330x_{n+1})$
5.	$x^2 - y^2 = 135424$	$\frac{1}{184}(2838x_{n+1} - 10y_{n+2})$, $\frac{1}{184\sqrt{33}}(66y_{n+2} - 16302x_{n+1})$
6.	$x^2 - y^2 = 286015744$	$\frac{1}{8456}(130482x_{n+1} - 10y_{n+3})$, $\frac{1}{8456\sqrt{33}}(66y_{n+3} - 74956x_{n+1})$
7.	$x^2 - y^2 = 4096$	$\frac{1}{32}(22714x_{n+2} - 494x_{n+3})$, $\frac{1}{32\sqrt{33}}(2838x_{n+3} - 130482x_{n+2})$
8.	$x^2 - y^2 = 135424$	$\frac{1}{184}(66x_{n+2} - 494y_{n+1})$, $\frac{1}{184\sqrt{33}}(2838y_{n+1} + 330x_{n+2})$
9.	$x^2 - y^2 = 256$	$\frac{1}{8}(2838x_{n+2} - 494y_{n+2})$, $\frac{1}{8\sqrt{33}}(2838y_{n+2} - 16302x_{n+2})$

2. Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table below:

S.No	Parabola	x, y
1.	$y^2 = 1472x - 8667136$	$\frac{1}{1472}(22714x_{n+1} - 10x_{n+3}) + 2$, $\frac{1}{1472\sqrt{33}}(66x_{n+3} - 130482x_{n+1})$
2.	$y^2 = 4x - 64$	$\frac{1}{16}(33x_{n+1} - 5y_{n+1}) + 2$, $\frac{33}{16}(y_{n+1} - 5x_{n+1})$

3.	$y^2 = 32x - 4096$	$\frac{1}{32}(494x_{n+1} - 10x_{n+2}) + 2,$ $\frac{1}{32\sqrt{33}}(66x_{n+2} - 2838x_{n+1})$
4.	$y^2 = 8x - 256$	$\frac{1}{8}(66x_{n+1} - 10y_{n+1}) + 2,$ $\frac{1}{8\sqrt{33}}(66y_{n+1} - 330x_{n+1})$
5.	$y^2 = 184x - 135424$	$\frac{1}{184}(2838x_{n+1} - 10y_{n+2}) + 2,$ $\frac{1}{184\sqrt{33}}(66y_{n+2} - 16302x_{n+1})$
6.	$y^2 = 8456x - 286015744$	$\frac{1}{8456}(130482x_{n+1} - 10y_{n+3}) + 2,$ $\frac{1}{8456\sqrt{33}}(66y_{n+3} - 749562x_{n+1})$
7.	$y^2 = 32x - 4096$	$\frac{1}{32}(22714x_{n+2} - 494x_{n+3}) + 2,$ $\frac{1}{32\sqrt{33}}(2838x_{n+3} - 130482x_{n+2})$
8.	$y^2 = 184x - 135424$	$\frac{1}{184}(66x_{n+2} - 494y_{n+1}) + 2,$ $\frac{1}{184\sqrt{33}}(2838y_{n+1} + 330x_{n+2})$
9.	$y^2 = 8x - 256$	$\frac{1}{8}(2838x_{n+2} - 494y_{n+2}) + 2,$ $\frac{1}{8\sqrt{33}}(2838y_{n+2} - 16302x_{n+2})$

3. Consider $p = x_{n+1} + y_{n+1}, q = x_{n+1}$. Observe that p>q>0.Treat p,q as the generators of the Pythagorean triangle T(α,β,γ),where $\alpha = 2pq, \beta = p^2 - q^2, \gamma = p^2 + q^2$.
Let A,P represent the area and perimeter of T(α,β,γ)
Then the following interesting relations are observed.

a) $2\alpha - 33\beta + 31\gamma = 16$

b) $35\beta - 33\gamma - \frac{8A}{P} = -16$

c) $6(2\alpha - \frac{4A}{P} + \beta)$ is a nasty number.

d) $3(\gamma-\frac{4A}{P}-\alpha+\beta)$ is a nasty number.

4. Conclusion:

In this paper, we have presented infinitely many integer solutions for the hyperbola represented by the negative Pell equation $y^2=33x^2-8$. As the binary quadratic Diophantine equations are rich in variety, one may search for the other choices of negative Pell equations and determine their integer solutions along with suitable properties.

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