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FOURIER SERIES AND ITS SOME APPLICATIONS

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Abstract:

The Fourier series, the founding principle behind the field of Fourier Analysis, is an infinite expansion of a function in terms of sines and cosines. In physics and engineering, expanding functions in terms of sines and cosines is useful because it allows one to more easily manipulate functions. In particular, the fields of electronics, quantum mechanics, and electrodynamics all make heavy use of the Fourier series.

Key words: Fourier Transform, Discrete Fourier Transform & Convolution Transforms

1. Introduction:

The Fourier series, the founding principle behind the field of Fourier analysis, is an infinite expansion of a function in terms of sines and cosines or imaginary exponentials. The series is defied in its imaginary exponential form as follows:

$$f(t) = \sum_{n=-\infty}^{\infty} A_{n}\, e^{-inx} \tag{1}$$

where A_n's are given by the expression

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\, dx \tag{2}$$

Thus, the Fourier series is an infinite superposition of imaginary exponentials with frequency terms that increase as n increases. Since sines and cosines (and in turn imaginary exponentials) form an orthogonal set[1], this series converges for any moderately well- behaved function $f(x)$ .Examples of the Fourier Series for different wave forms are given in figure I.

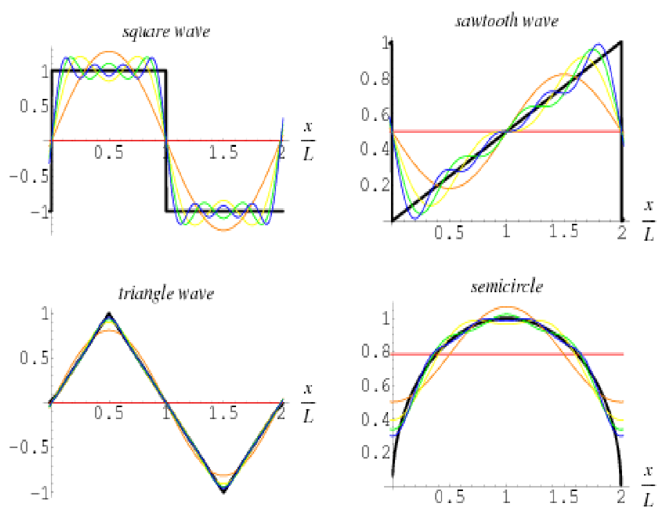
2. The Fast Fourier Transforms:

The Fourier series is only capable of analyzing the frequency components of certain, discrete frequencies (integers) of a given function. In order to study the case where the frequency components of the sine and cosine terms are continuous, the concept of the Fourier Transform must be introduced. The imaginary exponential form of the Fourier Transform is defined as follows:

$$H(\omega) = \int_{-\infty}^{\infty} h(t)e^{i\omega t}\, dt \tag{3}$$

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\, e^{i\omega t}\, dt \tag{4}$$

Fig. 1: Fourier series Examples



Here, the $H(\omega)$ fulfils the role of the A_n's in equations(1) and (2); it gives an indicator of "how much" a particular frequency oscillation contributes to the function $f(t)$. Assume that we have a equidistant, finite data set $h_k = h(t_k)$, $t_k = k\Delta$, and we are only interested in N equidistant, discreet frequencies in the range $-\omega_c$ to ω_c.We thus wish to examine the frequencies $\omega_n = \frac{2\pi n}{N\Delta}$,for $N = \frac{-N}{2}, \dots, \frac{N}{2}$,where we let Δ be the same Δ which defines[1, 2] our t_k's. We may provide an approximation to the Fourier Transform in this range by a Riemann sum as follows:

$$H(\omega_n) = \int_{-\infty}^{\infty} h(t)e^{it}\, dt$$
$$\approx \sum_{k=0}^{N-1} h_k\, e^{i\omega_n t_k} \Delta$$

Using our definition of t_k , this expression reduces to

$$H(w) \approx \Delta \sum_{k=0}^{N-1} h_k\, e^{ik\frac{2\pi n}{N}} \quad (5)$$

This equation is called the Discrete Fourier Transform (DFT) of the function $h(t)$. If we denote H_n as

$$H_n = \sum_{k=0}^{N-1} h_k\, e^{ik\frac{2\pi n}{N}} \quad (6)$$

The Fourier Transform,$H(\omega)$, may then be approximate reducing the expression

$$H(w) \approx \Delta\, H_n \quad (7)$$

Comparing equation (6) with the Fourier series given in equation (1), it is clear that this is a form of the Fourier series with non-integer frequency components. Currently, the most common and efficient method of numerically calculating the DFT is by using a class of algorithms called "Fast Fourier Transforms" (FFTs). The first known discovery of the FFT was by Gauss in 1805; however, that modern "rediscovery" of the FFT was done in 1942 by Danielson and Lanczos. They were able to show one may divide any DFT into a sum of two DFT's which each correspond to $\frac{N}{2} - 1$ points.

The proof of Danielson and Lanczos assertion is the following:

First, define W as the complex number

$$W = e^{2\pi i/N} \quad (8)$$

Equation (6) may be then be written

$$H_n = \sum_{k=0}^{N-1} w^{nk}\, h_k \quad (9)$$

Any DFT may then be written as follows:

$$\begin{aligned} H_n &= \sum_{k=0}^{N-1} e^{2\pi ink/N}\, h_n \\ &= \sum_{k=0}^{N/2-1} e^{2\pi i(2k)n/N} h_{2k} + \sum_{k=0}^{N/2-1} e^{2\pi i(2k+1)n/N}\, h_{2k+1} \\ &= \sum_{k=0}^{N/2-1} e^{2\pi i(2k)n/(N/2)}\, h_{2k} + W^k \sum_{k=0}^{N/2-1} e^{2\pi i(2k)n/(N/2)} h_{2k+1} \\ &= H_n^e + W^k H_n^O \end{aligned}$$

Here, H_n^e denotes the even terms of the sum (the ones corresponding to the index $2k$) and H_n^O denotes the odd terms (the ones corresponding to the index $(2k+1)$. The most useful part of this formula is that it can be used recursively, since each of these H_n^e and H_n^O terms may be independently expanded using the same algorithm, each time reducingthe number of calculations by a factor of 2. In fact, this class of FFT algorithm shrinks the computation time from $O(N^2)$operations to the much more manageable $O(N log_2 N)$operations. There are many different FFT algorithms; the one presented here is simply the most common one, known as a Cooley-Turkey FFT algorithm. There are other algorithms which can decrease computation time by 20 or 30 percent (so-called base-4 FFTs or base-8 FFTs). Most importantly, both classes of FFT algorithms are fast enough to embed into modern digital oscilloscopes and other such electronic equipment. Thus, FFTs have many modern applications, such as Spectrum Analyzers, Digital Signal Processors (DSPs), and the numerical computation arbitrary-size multiplication operations.

3. The Spectrum Analyzer:

An important instrument to any experimentalist is the spectrum analyzer. This instrument reads a signal (usually a voltage) and provides the operator with the Fourier coefficient which corresponds to each of the sine and cosine terms of the Fourier expansion of the signal. Suppose an instrument takes a time-domain signal, such as the amplitude of the output voltage of an instrument.

Let us call this signal V(t). Then the DFT of V(t) is

$$H_n = \sum_{k=0}^{N-1} v_k e^{ik\frac{2n}{N}} \quad (10)$$

We see that this equation is of the same form of equation (6), which means that the previously described methods of the FFT apply to the function. Thus, any digital oscilloscope that is sufficiently fast and equipped with a FFT algorithm is capable of providing the user with the frequency components of the source signal. Oscilloscopes which are equipped with the ability to FFT their inputs are termed "Digital Spectral Analyzers". Although they were once a separate piece of equipment for experimentalists, improvements in digital electronics has made it practical to merge the role of oscilloscopes with that of the Spectral Analyzer; it is quite common now that FFT algorithms come built into oscilloscopes.

Spectrum Analyzers have many uses in the laboratory, but one of the most common uses is for signal noise studies. As shown above, the FFT of the signal gives the amplitudes of the various oscillatory components of the input. After normalization, this allows for the experimentalist to determine what frequencies dominate their signal. For example, if you have a DC signal, you would expect the FFT to show only very low frequency oscillations (i.e., the largest amplitudes should correspond to $f \approx 0$). However, if you see a sharp peak of amplitudes around 60 Hz, you would know that something is feeding noise into your signal with a frequency of 60 Hz (for example, an AC leakage from your power source).

4. Digital Signal Processing:

We have already seen how the Fourier series allows experimentalists to identify sources of noise. It may also be used to eliminate sources of noise by introducing the idea of the Inverse Fast Fourier Transform (IFFT).In general, the goal of an Inverse Fourier Transform is to take the An (the ones that appear in equation (5) and use them to reconstruct the original function, f(t).Analytically, this is done by multiplying each A_n by$e^{2\pi k\frac{n}{N}}$ then taking the sum over all n. However, thisis an inefficient algorithm to use when the calculation must be done numerically. Just as there is a fast numerical algorithm for approximating the Fourier coefficients (the FFT), there is another efficient algorithm, called the IFFT, which is capable of calculating the Inverse Fourier Transform much faster than the brute-force method. In 1988, it was shown by Duhamel, Piron, and Etcheto[7] that the IFFT is simply

$$F^{-1}(x) = F(ix^*)^* \quad (11)$$

In other words, you can calculate the IFFT directly from the FFT; you simply ip the real and imaginary parts of the coefficients calculated by the original FFT. Thus, the IFFT algorithms are essentially the same as the FFT algorithms; all one must do is ip the numbers around at the beginning of the calculation.

Since the IFFT inherits all of the speed benefits of the FFT, it is also quite practical to use it in real time in the laboratory. One of the most common applications of the IFFT in the laboratory is to provide Digital Signal Processing (DSP). In general, the idea of DSP is to use configurable digital electronics to clean up, transform, or amplify a signal by first FFT'ing the signal, removing, shifting or damping the unwanted frequency components, and then transforming the signal back using the IFFT on the filtered signal. There are many advantages to doing DSP as opposed to doing analog signal processing. To begin with, practically speaking, you can have a much more complicated filtering function (the function that transforms the coefficients of the DFT) with DSP than analog signal processing. While it is fairly easy to make a single band pass, low pass, or high pass filter with capacitors, resistors, and inductors, it is relatively difficult and time consuming to implement anything more complicated than these three simple filters. Furthermore, even if a more [3, 4] complicated filter was implemented with analog electronics, it is difficult to make even small modifications to the filter (there are exceptions to this, such as FPGA's, but those are also more difficult to implement than simple software solution). DSP is not limited by either of these effects since the processing is (usually) done in software, which can be programmed to do whatever the user desires.

Probably the most important advantage that DSP has over analog signal processing is the fact that the processing may be done after the signal has been taken. In modern-day experiments, raw data is often recorded during the experiment and corrected for noise in software during the analysis step. If one filters the signal before-hand (with analog signal processing), it is possible that later in the experiment, the experimenter could find that they filtered out good signal. The only option in this case is for the experiment to be rerun. On the other hand, if the signal processing was done digitally, all that the experimenter has to do is edit their analysis code and rerun the analysis; this could save both time and money.

5. Analytical Applications:

The Fourier series also has many applications in mathematical analysis. Since it is a sum of multiple sines and cosines, it is easily differentiated and integrated, which often simplifies analysis of functions such as saw waves which are common signals in experimentation.

A. Discontinuous Functions:

The Fourier series also offers a simplified analytical approach to dealing with discontinuous functions. Dirichlet's Theorem states the following:

If f(x) is a periodic of period 2π, and if between $-\pi$ and π it is single-valued, has a finite number of maximum and minimum values, and a finite number of discontinuities, and If $\int_{-\pi}^{\pi} |f(x)|\, dx$ finite, then the Fourier series converges to f(x) at all the points where f(x) is continuous; at jumps, the Fourier series converges to the midpoint of the jump (This includes jumps that occur at $\pm\pi$forthe periodic function). In other words, nearly every function encountered in physics, both continuous and discontinuous, may be represented in terms of the Fourier series. This gives the Fourier series a distinct advantage over the Taylor Series expansion of a function; since the Taylor Series places much more stringent limits on convergence than the Fourier series does (continuity is a requirement, for example).

B. Convolutions:

The Convolution Theorem states the following:

$$F^{-1}(F[f]F[g]) = f * g \qquad (12)$$

Where $F[f]$ denotes the Fourier Transform of thefunction f. Since the Fourier Transform may be approximated by a Fourier series, FFT algorithms may be applied to the [5, 6] numerical calculation of the convolution; in fact, the FFT method is the preferred method of calculating convolutions which prevents the need for direct integration.

C. Generalized Fourier series:

The concept of the Fourier series may be generalized to any complete orthogonal system of functions. An orthogonal system" satisfies the following relation

$$\int \emptyset_m(x)\emptyset_n(x)w(x) = c_m\delta_{mn} \quad (13)$$

In a generalized Fourier series, we use these functions $\emptyset(x)$ as the expansion functions instead of sines and cosines (or imaginary exponentials). Then our expansion takes on the following form

$$f(x) = \sum_{n=0}^{\infty} a_n \emptyset_n\ (x) \quad (14)$$

One may then find the coefficients an in an analogous way that one finds the coefficients in the Fourier series:

$$\begin{aligned}\int f(x)\emptyset_n\ (x)w(x)dx &= \sum_{n=0}^{\infty} a_n \int \emptyset_m(x)\emptyset_n(x)w(x)dx \\ &= \sum_{n=0}^{\infty} a_n \int \emptyset_m(x)\emptyset_n\ (x)w(x)dx \\ &= \sum_{n=0}^{\infty} a_n c_m \delta_{mn} \\ &= a_n c_n\end{aligned}$$

Where c_n is the normalization constant given by the orthogonality relationship defined in (13). Equating the first and last parts leaves us with

$$\frac{1}{c_n}\int f(x)\emptyset_n(x)w(x)dx = a_n \quad (15)$$

This result is analogous to the result that was presented in equation (2), and can be used to derive these expressions. An example of another complete orthogonal system which can be used as the basis element for a Generalized Fourier Series is the set of Spherical Harmonics. The Spherical Harmonics provide the series that is analogous to the Fourier series, called the Laplace series, which is given by the expression

$$f(\theta, \emptyset) = \sum_{l=0}^{\infty}\sum_{m=0}^{l} A_l^m Y_l^m(\theta, \emptyset) \quad (16)$$

Functional expansions of this form are termed Generalized Fourier Series" since they utilize the orthogonality relationships of functional systems in the same way that the Fourier series does.

6. Conclusion:

The Fourier series is useful in many applications ranging from experimental instruments to rigorous mathematical analysis techniques. Thanks to modern developments in digital electronics, coupled with numerical algorithms such as the FFT, the Fourier series has become one of the most widely used and useful mathematical tools available to any scientist.

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