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REVERSE ORDER LAW FOR Con-s-k-EP MATRICES

Dr. B. K. N. Muthugobal* & R. Surendar**
* Assistant Professor of Mathematics, Bharathidasan University Constituent College, Nannilam, Tamil Nadu, India.
** Assistant Professor of Mathematics, Swami Dayananda College of Arts and Science, Manjakkudi, Tiruvarur, Tamilnadu, India.

Abstract:

Necessary and sufficient conditions for $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ are defined for a pair of Con-s-k-EP matrices A and B.

Keywords: Con-s-k-EP Matrix, Generalized Inverse & Reverse Order.

1. Introduction:

Let C_{nxn} be the space of nxn complex matrices of order n. let C_n be the space of all complex n tuples. For $A \epsilon C_{nxn}$. Let $\bar{A}$, A^T, A^*, A^S , $\bar{A}^S$, $A^{\dagger}$, $R(A)$, $N(A)$ and $\rho(A)$ denote the conjugate, transpose, conjugate transpose, secondary transpose, conjugate secondary transpose, Moore Penrose inverse range space, null space and rank of A respectively. A solution X of the equation $AXA = A$ is called generalized inverse of A and is denoted by A^{-}. If $A \epsilon C_{nxn}$ then the unique solution of the equations $AXA = A$, $XAX = X$, $[AX]^* = AX$, $(XA)^* = XA$ [6] is called the Moore-Penrose inverse of A and is denoted by $A^{\dagger}$. A matrix A is called Con-s-k-EP$_r$ if $\rho(A) = r$ and $N(A) = N(A^T VK)$ (or) $R(A) = R(KVA^T)$. Throughout this paper let k be the fixed product of disjoint transposition in $S_n = \{1,2,\ldots,n\}$ and k be the associated permutation matrix.

Let us define the function $k(x) = \left(x_{k(1)}, x_{k(2)}, \ldots, x_{k(n)}\right)$. A matrix $A = (a_{ij}) \epsilon C_{nxn}$ is s-k-symmetric if $a_{ij} = a_{n-k(j)+1, n-k(i)+1}$ for $i, j = 1,2,\ldots,n$. A matrix $A \epsilon C_{nxn}$ is said to be Con-s-k-EP if it satisfies the condition $A_x = 0 \Leftrightarrow A^s k(x) = 0$ or equivalently $N(A) = N(A^T VK)$. In addition to that A is con-s-k-EP $\Leftrightarrow$ KVA is con-EP or AVK is con-EP and A is con-s-k-EP $\Leftrightarrow$ A^T is con-s-k-EP$_r$ moreover A is said to be Con-s-k-EP$_r$ if A is con-s-k-EP and of rank r. For further properties of con-s-k-EP matrices one may refer [3].

For any two non singular matrices $A, B \in C_{n\times n}$, $(AB)^{-1} = B^{-1}A^{-1}$ holds. However, it is not true for generalized inverses of matrices. In general, $(AB)^{\dagger} \neq B^{\dagger}A^{\dagger}$, for any two matrices A and B. For example, $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $AB = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1]$, $(AB)^{\dagger} = [1]$ and $B^{\dagger}A^{\dagger} \neq (AB)^{\dagger}$ we say that the reverse order law holds for Moore Penrose inverse of the product of A and B, if $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$. It is well known that (p.181) [1], $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ if and only if $R(BB^T A^T) \subseteq R(A^T)$ and $R(A^T AB) \subseteq R(B)$.

2. Preliminaries:

Theorem 1: (p.24) [1]

For any $A \in C_{n\times n}$, the following hold.

(i) $R(A^{\dagger}) = R(A^*)$ *and* $N(A^{\dagger}) = N(A^*)$

(ii) $R(A) = R(B) \Leftrightarrow AA^{\dagger} = BB^{\dagger}$

Theorem 2: (p.24) [1]

For any $A \in C_{n\times n}$, the following hold:

(i) $(A^{\dagger})^{\dagger} = A$

(ii) $A^{\dagger} = A^{-1} \Leftrightarrow A$ is non singular.

(iii) $(A^{*})^{\dagger} = (A^{\dagger})^{*}$.

Theorem 3: (p.21) [6]

Let $A, B \in C_{n\times n}$. Then

(i) $N(A) \subseteq N(B) \Leftrightarrow R(B^{*}) \subseteq R(A^{*}) \Leftrightarrow B = BA^{-}A$ for all $A^{-} \in A\{1\}$

(ii) $N(A^{*}) \subseteq N(B^{*}) \Leftrightarrow R(B) \subseteq R(A) \Leftrightarrow B = AA^{-}B$, for every $A^{-} \in A\{1\}$.

Theorem 4: [2]

For an $n\times n$ matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ the following are equivalent:

(i) $\rho(M) = \rho(A)$

(ii) $N(A) \subseteq N(C), N(A^*) \subseteq N(B^*)$ and $(M/A) = 0$.

Theorem 5: (Theorem 1[9] and [5])

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Then $M^{\dagger} = \begin{pmatrix} A^{\dagger} + A^{\dagger}B(M/A)^{\dagger}CA & -A^{\dagger}B(M/A)^{\dagger} \\ -(M/A)^{\dagger}CA^{\dagger} & (M/A)^{\dagger} \end{pmatrix}$

$\Leftrightarrow N(A) \subseteq N(C), N(A^{*}) \subseteq N(B^{*}), N((M/A)^{*}) \subseteq N(C^{*})$ *and* $N(M/A) \subseteq N(B)$.

Also, $M^{\dagger} = \begin{pmatrix} (M/D)^{\dagger} & -A^{\dagger}B(M/A)^{\dagger} \\ -D^{\dagger}C(M/D)^{\dagger} & (M/A)^{\dagger} \end{pmatrix}$

$\Leftrightarrow N(A) \subseteq N(C), N(A^{*}) \subseteq N(B^{*}), N((M/A)^{*}) \subseteq N(C^{*}), N(M/A) \subseteq N(B)$ and $\Leftrightarrow N(D) \subseteq N(B), N(D^{*}) \subseteq N(C^{*}), N((M/D)^{*}) \subseteq N(B^{*}), N(M/D) \subseteq N(C)$. When $\rho(M) = \rho(A)$, then $M = \begin{pmatrix} A & B \\ C & CA^{-}B \end{pmatrix}$ and $M = \begin{pmatrix} A^{*}PA^{*} & A^{*}PC^{*} \\ B^{*}PA^{*} & B^{*}PC^{*} \end{pmatrix}$ where $p = (AA^{*} + BB^{*})^{-} A (A^{*}A + C^{*}C)^{-}$.

3. Reverse Order Law for Con-s-k-EP Matrices:

Theorem 6:

If A and B are con-s-k-EP$_r$ matrices with $R(A) = R(B^{T})$ then $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

Proof:

Since A is con-s-k-EP$_r$, $R(A) = R(KVA^{T})$.

$\Rightarrow R(B^{T}) = R(KVA^{T})$ (by hypothesis)

$\Rightarrow R(KVB) = R(KVA^{T})$ (since B is con-s-k-EP$_r$)

$\Rightarrow R(B) = R(A^{T})$ (since $R(KVA) = R(KVB)$)

$\Rightarrow R(A) = R(B)$

$\Rightarrow R(B) = R(A^{\dagger})$ (by Theorem 1)

That is given $x \in C_n$, there exists a $y \in C_n$ such that $Bx = A^{\dagger}y$.

Now, $Bx = A^{\dagger}y \Rightarrow B^{\dagger}A^{\dagger}ABx = B^{\dagger}(A^{\dagger}AA^{\dagger})y$

$\Rightarrow B^{\dagger}A^{\dagger}ABx = B^{\dagger}A^{\dagger}y$

$\Rightarrow B^{\dagger}A^{\dagger}ABx = B^{\dagger}Bx$

Since $B^{\dagger}B$ is symmetric, it follows that, $B^{\dagger}A^{\dagger}AB$ is symmetric.

Similarly, $A^{\dagger}y = Bx \Rightarrow ABB^{\dagger}A^{\dagger}y = ABB^{\dagger}Bx$

$$\Rightarrow ABB^{\dagger}A^{\dagger}y = ABx$$

$$\Rightarrow ABB^{\dagger}A^{\dagger}y = AA^{\dagger}y$$

Since, $AA^{\dagger}$ is symmetric, it follows that $ABB^{\dagger}A^{\dagger}$ is symmetric. Further, by Theorem 2, $R(A) = R(B) \Rightarrow AA^{\dagger} = BB^{\dagger}$

$$R(A^{\dagger}) = R(B) \Rightarrow A^{\dagger}(A^{\dagger})^{\dagger} = BB^{\dagger}$$

$$\Rightarrow A^{\dagger}A = BB^{\dagger}$$

Hence, $(AB)(B^{\dagger}A^{\dagger})(AB) = ABB^{\dagger}(A^{\dagger}A)B$

$$= ABB^{\dagger}(BB^{\dagger})B$$

$$= ABB^{\dagger}B$$

$$= AB$$

$$(B^{\dagger}A^{\dagger})(AB)(B^{\dagger}A^{\dagger}) = B^{\dagger}(A^{\dagger}A)(BB^{\dagger})A^{\dagger}$$

$$= B^{\dagger}A^{\dagger}$$

Thus, $B^{\dagger}A^{\dagger}$ satisfies the defining equations of the Moore-Penrose inverse, that is, $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

Remark 1:

In the above Theorem, the condition that $R(A) = R(B^T)$ is essential.

Example 1:

Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for k = (1,2)(3), the associated Permutation matrix $K = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, A and B are con-s-k-EP_1 matrices.

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (AB)^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{\dagger} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$B^{\dagger}A^{\dagger} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Here $R(A) = R(B^T)$. Thus $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$.

Example 2:

Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ for k = (1,2)(3), the associated permutation matrix $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ and $V=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$.

A and B are con-s-k-EP$_r$ matrices.

$$AB=\begin{pmatrix}1&0&0\\1&0&0\\1&0&0\end{pmatrix},\quad \rho(AB)=1. \qquad R(A)\neq R(B^T)$$

$$A^{\dagger}=\frac{1}{9}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \quad \text{and} \quad B^{\dagger}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$$

$$B^{\dagger}A^{\dagger}=\frac{1}{9}\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}, \qquad (AB)^{\dagger}=\frac{1}{3}\begin{pmatrix}1&1&1\\0&0&0\\0&0&0\end{pmatrix}$$

Thus $(AB)^{\dagger}\neq B^{\dagger}A^{\dagger}$.

Remark 2:

The converse of the Theorem 6 need not be true in general.

For let $A=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$, $B=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}$ for $k = k_1k_2$, the associated permutation matrix $K=\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}$ and $V=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$. A and B are con-s-k-EP$_1$ matrices.

$$AB=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix},\ A^{\dagger}=\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix},\ B^{\dagger}=\begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}$$

$$(AB)^{\dagger}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}, \qquad B^{\dagger}A^{\dagger}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$

$(AB)^{\dagger}=B^{\dagger}A^{\dagger}$. But $R(A)\neq R(B^T)$.

Next to establish the validity of the converse of the Theorem 6 under certain conditions first let us prove the following lemma.

Lemma 1: Let $A=\begin{pmatrix}E&F\\G&H\end{pmatrix}$ be a con-s-k-EP$_r$ matrix with $k = k_1k_2$, $K_1\mathcal{V}G$ is an $n\times n$ matrix and $\begin{bmatrix}K_1\mathcal{V}G & K_1\mathcal{V}H\end{bmatrix}$ has rank r, Then $K_1\mathcal{V}G$ is non singular. Moreover there is an $n\times n$ matrix L such that $K\mathcal{V}A=\begin{pmatrix} K_1\mathcal{V}G & K_1\mathcal{V}GL^T \\ LK_1\mathcal{V}G & LK_1\mathcal{V}GL^T \end{pmatrix}$.

Proof:

$K_1\mathcal{V}G$ is an non singular follows from Lemma (2.19) [4]. Since $\rho(KVA)=\rho(K_1\mathcal{V}G)=r$ by Theorem 3 and Theorem 4 we have $K_2\mathcal{V}E=LK_1\mathcal{V}G,\ K_1\mathcal{V}H=K_1\mathcal{V}GQ$ where L and Q are $n\times n$ matrices. Therefore, $KVA=\begin{pmatrix} K_1\mathcal{V}G & K_1\mathcal{V}GQ \\ LK_1\mathcal{V}G & LK_1\mathcal{V}GQ \end{pmatrix}$.

Now, let $C=\begin{pmatrix} I_n & 0 \\ -L & I_n \end{pmatrix}$ and consider,

$$CKVAC^T=\begin{pmatrix} I_n & 0 \\ -L & I_n \end{pmatrix}\begin{pmatrix} K_1\mathcal{V}G & K_1\mathcal{V}GQ \\ LK_1\mathcal{V}G & LK_1\mathcal{V}GQ \end{pmatrix}\begin{pmatrix} I_n & -L^T \\ 0 & I_n \end{pmatrix}$$

$$=\begin{pmatrix} K_1\mathcal{V}G & K_1\mathcal{V}GQ \\ 0 & 0 \end{pmatrix}\begin{pmatrix} I_n & -L^T \\ 0 & I_n \end{pmatrix}$$

$$=\begin{pmatrix} K_1\mathcal{V}G & -K_1\mathcal{V}GL^T+K_1\mathcal{V}GQ \\ 0 & 0 \end{pmatrix}$$ which is again con-EP$_r$.

From $N(K\mathcal{V}A)=N(CK\mathcal{V}AC^T)$ it follows that,

$$K_1\mathcal{V}GQ-K_1\mathcal{V}GL^T=0\Rightarrow K_1\mathcal{V}GQ=K_1\mathcal{V}GL^T\Rightarrow Q=L^T.$$

Thus $K\mathcal{V}A=\begin{pmatrix} K_1\mathcal{V}G & K_1\mathcal{V}GL^T \\ LK_1\mathcal{V}G & LK_1\mathcal{V}GL^T \end{pmatrix}$.

Theorem 7:

If A, B are con-s-k-EP$_r$ matrices, $\rho(AB)=r$ and $(AB)^{\dagger}=B^{\dagger}A^{\dagger}$ then $R(A)=R(B^T)$.

Proof:

Since A is con-s-k-EP$_r$, there exists unitary matrix U and $2r\times 2r$ non singular matrix D such that $A=KVU\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}U^T$. That is, $U^TKVAU=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$.

Let $$U^TBVKU=\begin{pmatrix} K_1\mathcal{V}B_2 & K_2\mathcal{V}B_1 \\ K_1\mathcal{V}B_4 & K_2\mathcal{V}B_3 \end{pmatrix}$$

$$U^TKVAUU^TBVKU=\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} K_1\mathcal{V}B_2 & K_2\mathcal{V}B_1 \\ K_1\mathcal{V}B_4 & K_2\mathcal{V}B_3 \end{pmatrix}$$

$$U^TKVABVKU=\begin{pmatrix} DK_1\mathcal{V}B_2 & DK_2\mathcal{V}B_1 \\ 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} D & 0 \\ 0 & I_n \end{pmatrix}\begin{pmatrix} K_1\mathcal{V}B_2 & K_2\mathcal{V}B_1 \\ 0 & 0 \end{pmatrix}$$ has rank r

and $U^T BVKUU^T KVAU = \begin{pmatrix} K_1 \mathcal{V} B_2 & K_2 \mathcal{V} B_1 \\ K_1 \mathcal{V} B_4 & K_2 \mathcal{V} B_3 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$

$$U^T BAU = \begin{pmatrix} K_1 \mathcal{V} B_2 D & 0 \\ K_1 \mathcal{V} B_4 D & 0 \end{pmatrix}$$

$$= \begin{pmatrix} K_1 \mathcal{V} B_2 & 0 \\ K_1 \mathcal{V} B_4 & 0 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & I_n \end{pmatrix} \text{ has rank } r.$$

It follows that, $\begin{pmatrix} K_1 \mathcal{V} B_2 & K_2 \mathcal{V} B_1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} K_1 \mathcal{V} B_2 & 0 \\ K_1 \mathcal{V} B_4 & 0 \end{pmatrix}$ have rank r,

So that B_2 is non singular. By Lemma 1 $U^T BVKU = \begin{pmatrix} K_1 \mathcal{V} B_2 & K_1 \mathcal{V} B_1 L^T \\ LK_1 \mathcal{V} B_2 & LK_1 \mathcal{V} B_2 L^T \end{pmatrix}$ with $\rho(U^T KVBU) = \rho(K_1 \mathcal{V} B_2) = r$. By using Theorem 5 of Penrose representation for the generalized inverse we get,

$$(U^T BVKU)^\dagger = \begin{pmatrix} (K_1 \mathcal{V} B_2)^T P (K_1 \mathcal{V} B_2)^T & (K_1 \mathcal{V} B_2)^T P (K_1 \mathcal{V} B_2)^T L^T \\ L (K_1 \mathcal{V} B_2)^T P (K_1 \mathcal{V} B_2)^T & L (K_1 \mathcal{V} B_2)^T P (K_1 \mathcal{V} B_2)^T L^T \end{pmatrix}$$

Where, $P = [(K_1 \mathcal{V} B_2)(K_1 \mathcal{V} B_2)^T + (K_1 \mathcal{V} B_2) L^T L (K_1 \mathcal{V} B_2)^T]^{-1}$
$(K_1 \mathcal{V} B_2)[(K_1 \mathcal{V} B_2)^T (K_1 \mathcal{V} B_2) + (K_1 \mathcal{V} B_2)^T L^T L (K_1 \mathcal{V} B_2)]^{-1}$.

Therefore, $U^T (BVK)^\dagger U = \begin{pmatrix} Q & QL^T \\ LQ & LQL \end{pmatrix}$,

where $Q = (I + L^T L)^{-1} (K_1 v B_2)^{-1} (I + L^T L)^{-1}$

Also, $U^T (KVA)^\dagger U = (U^T KVAU)^\dagger \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}$.

Now, $U^T KVABVK = (U^T KVABVKU)(U^T KVABVKU)^{-1}(U^T KVABVKU)$

$$= (U^T KVABVKU) U^T KV (AB)^\dagger VKU (U^T KVABVKU)$$
$$= (U^T KVABVKU) U^T KVB^\dagger A^\dagger VKU (U^T KVABVKU)$$
$$= (U^T KVABVKU) U^T KVB^\dagger UU^T A^\dagger VKU (U^T KVABVKU)$$
$$= (U^T KVABVKU)(U^T (BVK)^\dagger U)(U^T (KVA)^\dagger U)(U^T KVABVKU)$$

Therefore,

$$\begin{pmatrix} DK_1 \mathcal{V} B_2 & DK_1 \mathcal{V} B_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} DK_1 \mathcal{V} B_2 & DK_1 \mathcal{V} B_1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} Q & QL^T \\ LQ & LQL^T \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} DK_1 \mathcal{V} B_2 & DK_1 \mathcal{V} B_1 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} DK_1 \mathcal{V} B_2 Q + DK_1 \mathcal{V} B_1 LQ & DK_1 \mathcal{V} B_2 QL^T + DK_1 \mathcal{V} B_1 LQL^T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} DK_1 \mathcal{V} B_2 & DK_1 \mathcal{V} B_1 \\ 0 & 0 \end{pmatrix}$$

$$=\begin{pmatrix}(DK_1\mathcal{V}B_2Q+DK_1\mathcal{V}B_1LQ)D^{-1} & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}DK_1\mathcal{V}B_2 & DK_1\mathcal{V}B_1\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}(DK_1\mathcal{V}B_2Q+DK_1\mathcal{V}B_1LQ)D^{-1}D(K_1\mathcal{V}B_2) & 0\\ 0 & 0\end{pmatrix}$$

$$=\begin{pmatrix}(DK_1\mathcal{V}B_2Q+DK_1\mathcal{V}B_1LQ)(K_1\mathcal{V}B_2) & 0\\ 0 & 0\end{pmatrix}$$

we get,

$$DK_1\mathcal{V}B_2=(DK_1\mathcal{V}B_2Q+DK_1\mathcal{V}B_1LQ)(K_1\mathcal{V}B_2)$$
$$=DK_1\mathcal{V}B_2(I+B_2^{-1}B_1L)Q(K_1\mathcal{V}B_2)$$

Since $K_1\mathcal{V}B_1=K_1\mathcal{V}B_2L^T$, $B_1=B_2L^T$ we have,

$$DK_1\mathcal{V}B_2=DK_1\mathcal{V}B_2(I+B_2^{-1}B_2L^TL)Q(K_1\mathcal{V}B_2)$$
$$\Rightarrow DK_1\mathcal{V}B_2=DK_1\mathcal{V}B_2(I+L^TL)Q(K_1\mathcal{V}B_2)$$
$$\Rightarrow QK_1\mathcal{V}B_2=(I+L^TL)^{-1}$$

Substituting this in the definition of Q we get $QK_1\mathcal{V}B_2=I$.

Therefore, $(I+L^TL)=(QK_1\mathcal{V}B_2)^{-1}=I$. Thus $L^TL=0\Rightarrow L=0$.

Therefore, $U^TBVKU=\begin{pmatrix}K_1\mathcal{V}B_2 & 0\\ 0 & 0\end{pmatrix}\Rightarrow U^TKVB^TU=\begin{pmatrix}B_2^T\mathcal{V}K_1 & 0\\ 0 & 0\end{pmatrix}$.

Also, $U^TKVAU=\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}$.

Since, D and $B_2^T\mathcal{V}K_1$ are $r\times r$ non singular matrices we have, $R(D)=R(B_2^T\mathcal{V}K_1)$.

$$\Rightarrow R\left[\begin{pmatrix}D & 0\\ 0 & 0\end{pmatrix}\right]=R\left[\begin{pmatrix}B_2^T\mathcal{V}K_1 & 0\\ 0 & 0\end{pmatrix}\right]$$
$$\Rightarrow R(U^TKVAU)=R(U^TKVB^TU)$$
$$\Rightarrow R(KVA)=R(KVB^T)$$
$$\Rightarrow R(A)=R(B^T)$$

Corollary 1:

If A, B are con-s-k-EP$_r$ matrices and $\rho(AB)=r$ then $(AB)^\dagger=B^\dagger A^\dagger \Leftrightarrow R(A)=R(B^T)$.

Theorem 8:

Let $M=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$ and $L=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$ be con-s-k-EP$_r$ matrices and $\rho(ML)=r$ then $(ML)^\dagger=L^\dagger M^\dagger \Leftrightarrow (CR)^\dagger=R^\dagger C^\dagger$ and $AC^\dagger\mathcal{V}K_1=K_2\mathcal{V}PR^\dagger \Leftrightarrow (CR)^\dagger=R^\dagger C^\dagger$ and $C^\dagger D\mathcal{V}K_2=K_1\mathcal{V}R^\dagger S$.

Proof:

Since $\rho(ML)=r$, using Theorem 7 and Theorem 6 and Theorem (2.24) of [4]

$(ML)^{\dagger} = L^{\dagger}M^{\dagger} \Leftrightarrow ML$ is con-s-k-EP$_r$

$\Leftrightarrow CR$ is con-s-k-EP$_r$ and $AC^{\dagger}\mathcal{V}K_1 = K_2\mathcal{V}PR^{\dagger}$

$\Leftrightarrow (CR)^{\dagger} = R^{\dagger}C^{\dagger}$ and $AC^{\dagger}\mathcal{V}K_1 = K_2\mathcal{V}PR^{\dagger}$

$\Leftrightarrow (CR)^{\dagger} = R^{\dagger}C^{\dagger}$ and $C^{\dagger}D\mathcal{V}K_2 = K_1\mathcal{V}R^{\dagger}S$.

Remark 3:

Theorem 8 fails if we relax the condition on the rank of ML.

For example let $M = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$, $L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and let

$K = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $V = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. Here $\rho(M) = 3$ and $\rho(L) = 3$.

But $\rho(ML) = 2$ and M are L con-s-k-EP$_3$ and $(ML)^{\dagger} = L^{\dagger}M^{\dagger}$, but ML is not con-s-k-EP. That is, the hypothesis that $\rho(ML) = r$ cannot be dropped.

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