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ON INTERESTING INTEGER PAIRS

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Abstract:

We search for two non-zero distinct positive integers a_0, a_1 *such that,*

$$(i): 2a_0 = \alpha^2, 2a_1 = \gamma^2, a_0 + a_1 = \beta^3$$

$$(ii): 2a_0 = \alpha^3, 2a_1 = \gamma^3, a_0 + a_1 = \beta^2$$

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Introduction:

Number Theory is a vast and fascinating field of mathematics concerned with the properties of numbers in general, and integers in particular as well as the wider classes of problems that arise from their study. The study of Number Theory is very important because all other branches depend on this branch for their final results. One of the oldest branches of Number Theory is the Diophantine equations [1-6]. Diophantine problems have fewer equations than unknown variables and involve finding integers that work correctly for all equations. In fact, Diophantine problems dominated most of the celebrated unsolved mathematical problems. Certain Diophantine problems come from physical problems or from immediate mathematical generalizations and others come from geometry in a variety of ways. Certain Diophantine problems are neither trivial nor difficult to analyze [7, 8].

In this communication, we search for two non-zero distinct positive integers a_0, a_1 such that,

$$(i): 2a_0 = \alpha^2, 2a_1 = \gamma^2, a_0 + a_1 = \beta^3$$

$$(ii): 2a_0 = \alpha^3, 2a_1 = \gamma^3, a_0 + a_1 = \beta^2$$

Method of Analysis:

Section A:

Let a_0, a_1 be any two non-zero distinct positive integers such that

$$2a_0 = \alpha^2, 2a_1 = \gamma^2, a_0 + a_1 = \beta^3 \quad (1)$$

Eliminating a_0, a_1 in the system of equation (1), we have

$$\gamma^2 + \alpha^2 = 2\beta^3 \quad (2)$$

Assume $\beta = a^2 + b^2$ (3)

Write (2) as

$$2 = (1+i)(1-i) \quad (4)$$

Substituting (3) and (4) in (2) and employing the method of factorization, define

$$(\gamma + i\alpha) = (1+i)(a+ib)^3$$

Equating the real and imaginary parts, we have

$$\gamma = a^3 - 3ab^2 - 3a^2b + b^3$$

$$\alpha = a^3 + 3a^2b - 3ab^2 - b^3$$

Then $a_0 = \alpha^2/2 = \left(\dfrac{(a^3+3a^2b-3ab^2-b^3)^2}{2}\right)$ (5)

$$a_1 = \gamma^2/2 = \left(\frac{(a^3-3ab^2-3a^2b+b^3)^2}{2}\right) \quad (6)$$

As the values of a_0, a_1 are to be in integer, it is possible to choose the parameters a, b in (5) & (6), so that a_0 & a_1 are integers. It is observed that a_0 & a_1 are integers when either a & b are of different parity or the same parity. A few examples are given below:

a	**b**	a_0	a_1	$2a_0$	$2a_1$	a_0+a_1
1	3	968	32	44^2	8^2	$(10)^3$
2	4	5408	2592	104^2	72^2	$(20)^3$
3	8	313632	800	792^2	40^2	$(68)^3$
5	7	91592	313632	428^2	792^2	$(74)^3$

It is worth to note that, in addition to (4), 2 may also be represented as follows:

$$2 = \left(\frac{(7+i)(7-i)}{25}\right)$$

$$2 = \left(\frac{(1+7i)(1-7i)}{25}\right)$$

Following the procedure as presented above, one obtains two more sets of distinct pairs satisfying (1). After performing a few calculations, the respective sets of pairs satisfying (1) are given below

Set 1:

$$a_0 = (525a^2b + 25a^3 - 175b^3 - 75a^2b)^2/2$$

$$a_1 = (25b^3 - 75ba^2 - 525a^2b + 175a^3)^2/2$$

Where a, b are both same parity.

Set 2:

$$a_0 = (75a^2b - 25b^3 + 175a^3 - 525ab^2)^2/2$$

$$a_1 = (25a^3 - 75b^2a - 525a^2b + 175b^3)^2/2$$

Where a, b are both same parity.

Section B:

Let a_0, a_1 be two non-zero distinct positive integers such that,

$$2a_0 = \alpha^3, 2a_1 = \gamma^3, a_0 + a_1 = \beta^2 \quad (7)$$

Eliminating a_0, a_1 in (7) we get,

$$\gamma^3 + \alpha^3 = 2\beta^2 \quad (8)$$

Replacing α by $2\alpha_1$ and γ by $2\gamma_1$ and β by $2\beta_1$ in (8), it is written as

$$\alpha_1^3 + \gamma_1^3 = \beta_1^2 \quad (9)$$

The above equation is satisfied by

$$\alpha_1 = m(m^3+n^3); \gamma_1 = n(m^3+n^3); \beta_1 = (m^3+n^3)^2$$

Thus,

$$\alpha = 2m(m^3+n^3);\gamma = 2n(m^3+n^3);\beta = 2(m^3+n^3)^2$$

Employing (7), we get

$$a_0 = \alpha^3/2 = 4m^3(m^3+n^3)^3$$

$$a_1 = \gamma^3/2 = 4n^3(m^3+n^3)^3$$

Note that $a_0 + a_1 = \beta^2$

A few numerical examples are given below:

m	n	a_0	a_1	$2a_0$	$2a_1$	a_0+a_1
1	2	2916	23328	18^3	216^3	162^2
2	4	11943936	95551488	288^3	13824^3	10368^2
1	3	87808	2370816	56^3	168^3	1568^2

Also, the introduction of the transformations

$$\alpha_1 = u+v, \gamma_1 = u-v, \beta_1 = 2u^2$$

in (9) leads to $v^2 = u^2(2u-1)/3$

Since our interest is on finding integer solutions, it is possible to choose 'u' so that the R.H.S of the above equation is a perfect square. Thus $u = 6k^2 - 6k + 2$ gives $v = (2k-1)(6k^2 - 6k + 2)$

Thus, in this case, the corresponding values of a_0 & a_1 satisfying (7) are given by

$$a_0 = 256k^3(3k^2-3k+1)^3$$

$$a_1 = 256(1-k)^3(3k^2-3k+1)^3$$

A few numerical examples are given below:

k	a_0	a_1	$2a_0$	$2a_1$	a_0+a_1
2	702464	87808	112^3	56^3	784^2
3	829898752	350113536	456^3	304^3	21904^2
4	47409408	14047232	1184^3	888^3	5776^2

Conclusion:

In this paper, we have obtained infinitely many pairs of integers (a_0, a_1) such that $2a_0, 2a_1, \ldots\ldots$as diophantine problems are rich in variety, one may attempt to find other choices of problems with their solutions.

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