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UNIFORM CONVERGENCE IN SEQUENCES AND SERIES OF FUNCTION

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Abstract:

Suppose $\{f_n\}, n = 1,2,3,\ldots$ *that is a sequence of function defined on a set E, and suppose that the sequence of number* $\{f_n(x)\}$ *converges for every* $x \in E$, *we define the function f by*

$$f(x) = \lim_{n\to\infty} f(x) \quad x \in E$$

Key Words: Integers, Converges, Uniform Converges & Sequence

Introduction:

Under the function we have to derive the following simple criterion for the uniform convergence of a series is very useful. The name comes from the letter traditionally used to denote the constants, or major ants that bound the functions in the series

Notation:

$\{f_n(x)\} \to$ Sequence of function
$E \to$ Set of function
$N \to$ Integers
$n \to$ Positive integers

Definition 1:

Let $\{f_n(x)\}$ be a sequence of function on E, sequence of function $\{f_n(x)\}$ is said to be convergence uniformly to f on E, If for any $\in > 0$, there exists an integer N (Depending on $\in$ only) such that [1]

$$|f_n(x) - f(x)| < \in \qquad \text{for all n>N} \quad x \in E$$

If the sequence of function $\{f_n(x)\}$ converges to f uniformly to f on E, We write $f_n \to f$ uniformly on E

Definition 2:

If $f_n \to 0$ uniformly on E and $\in > 0$, then there exists N>0 such that for all n>N and all $x \in E$

$$|f_n(x)| \le \in$$

This implies that for all n>N

$$\sup_{x\in E} |f_n(x)| \le \in \quad \text{and}$$

$$\text{Hence} \quad \lim_{n\to\infty} \sup_{x\in E} |f_n(x)| \le \in$$

Since $\in > 0$ is an arbitrary positive integer, such that [2]

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x)| = 0$$

Theorem: 1

$f_n \to f$ Uniformly on E if and only if $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$

Proof:

Let sequence uniformly Converges on E and given , and given $\in > 0$ there exists an integer N>0 such that for all n>N and all $x \in E$ [7]

$$\left| f_n(x) - f(x) \right| < \in \quad \text{for all n>N}$$

$\sup_{x\in E} \left| f_n(x) \le \in \right|$ and

Hence $\lim_{n\to\infty} \sup_{x\in E} \left| f_n(x) \right| \le \in$

Since $\in > 0$ is an arbitrary positive integer, such that [2]

$$\lim_{n\to\infty} \sup_{x\in E} \left| f_n(x) \right| = 0 \qquad (1)$$

Conversely,

Let the given condition is (1) is holds, for a given, let us choose an integer N such that,[2]

$$\left| f_n(x) - f(x) \right| < \in \qquad \text{for } n \ge N$$

$$f_n \to f \text{ Uniformly on E}$$

Theorem: 2

A sequence converges uniformly on E If and only if for a given $\in > 0$, there exists N>0 such that for all $n \ge m \ge N$ and for all $x \in E$,

$$\left| f_n(x) - f_m(x) \right| < \in \qquad (2)$$

This is said to be Cauchy criterion for uniform convergence.

Proof:

Let the sequence of function $\{f_m(x)\}$ be a uniformly convergence on E, and for any $\in > 0$, there exists an integer N_1 such that,[3],[4]

$$\left| f_m(x) - f(x) \right| < \frac{\in}{2}, \text{ for all } m>N_1 \text{ and}$$

Let the sequence of function $\{f_n(x)\}$ be a uniformly convergence on E, and for any $\in > 0$, there exists an integer N_2 such that,[5]

$$\left| f_n(x) - f(x) \right| < \frac{\in}{2}, \text{ for all m>N}$$

$N = \max(N_1, N_2)$

$$\left| f_n(x) - f_m(x) \right| \le \left| f_n(x) - f(x) \right| + \left| f_m(x) - f(x) \right|$$

$$\le \frac{\in}{2} + \frac{\in}{2}$$

$$< \in \qquad \text{For all } N_1, N_2 > N$$

Conversely,

Let the given Condition hold, by Cauchy‘s general principle of Convergence for each $x \in E$ to a limit, say *f* and so the sequence convergence point wise to f

For a given $\in > 0$.

Let us choose an integer N such that (1) holds,

Fix n and $m \to \infty$ in (2). Since we get

$$\left| f(x) - f_n(x) \right| < \in \quad \text{For all } n \ge N \ \ x \in E$$

Which proves that $f_n(x) \to f(x)$ uniformly on E.

Theorem: 3

Suppose that $f_n(x)$ is a sequence of function defined on E, and suppose

$$|f_n(x)| \leq M_n \qquad (x \in E, n = 1,2,3,.....)$$

Then converges uniformly on E if $\sum M_n$ is said to be Weierstrass M-test

Proof:

Lt $\sum f_n$ is uniformly Cauchy if $\sum M_n$ is Cauchy

Let ϵ > 0 be given. The Cauchy condition for the convergence of a real series implies that there exists N ∈ N such that [4]

$$\sum_{k=m+1}^{n} M_k < \in \text{ for all } n > m > N.$$

Then for all x ∈ A and all n > m > N, we have [6]

$$\left|\sum_{k=m+1}^{n} f_k(x)\right| \leq \sum_{k=m+1}^{n} |f_k(x)|$$

$$\leq \sum_{k=m+1}^{n} M_k$$

$$\leq \in$$

Thus, $\sum f_n$ satisfies the uniform Cauchy condition,

Hence it is converges uniformly.

Conclusion:

In this paper we derived some important theorem based on uniform convergence of the function.

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