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# PYTHAGOREAN TRIANGLE WITH HYPOTANEOUS - $4\left(\frac{\text{Area}}{\text{Perimeter}}\right) = (4k^2 - 4k + 3)\alpha^2$

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**Abstract:**

*Patterns of Pythagorean triangle, where in each of which, Hypotaneous-* $4\left(\frac{\text{Area}}{\text{Perimeter}}\right) = (4k^2 - 4k + 3)\alpha^2$ *are obtained. Also, a few Diophantine m tuples, (m=3, 4) with suitable property are constructed through the linear combination among the generators of the Pythagorean triangle.*

**Index Terms:** Area/Perimeter, Pythagorean Triangle, Square Integer, Diophantine 3-Tuple, 4-Tuple

**Mathematical Subject Classification: 11D09, 11D99**

**Introduction:**

The Pythagorean numbers play a significant role in the theory of higher arithmetic as they come in the majority of indeterminate problems; had a marvelous effect on a credulous people and always occupy a remarkable position due to unquestioned historical importance. The method of obtaining three non-zero integers x, y and z under certain relations satisfying the relation $x^2 + y^2 = z^2$ has been a matter of interest to various Mathematicians [1-6]. In [7-21], special Pythagorean problems are studied. In this communication, we search for patterns of Pythagorean triangles, where, in each of which the Hypotaneous- $4\left(\frac{\text{Area}}{\text{Perimeter}}\right) = (4k^2 - 4k + 3)\alpha^2$ are obtained. Also, a few Diophantine m tuples, (m=3, 4) with suitable property are constructed through the linear combination among the generators of the Pythagorean triangle.

**Method Of Analysis:**

It is well known that the Pythagorean equation $\mathrm{T}(x, y, z) : x^2 + y^2 = z^2$ is satisfied by $x = 2pq, y = p^2 - q^2, z = p^2 + q^2, (p > q > 0)$ (1)

Let A,P represents the area and perimeter of the Pythagorean triangle $\mathrm{T}(x, y, z)$

The assumption $z - 4\left(\frac{A}{P}\right) = (4k^2 - 4k + 3)\alpha^2, (k, \alpha > 0)$ (2)

leads to the equation $(p - q)^2 + 2q^2 = (4k^2 - 4k + 3)\alpha^2$ (3)

Consider $\alpha = a^2 + 2b^2$ (4)

Using (4) in (3), it is written in the factorizable form as,

$$(p - q + i\sqrt{2}q)(p - q - i\sqrt{2}q) = (2k - 1 + i\sqrt{2})(2k - 1 - i\sqrt{2})(a + i\sqrt{2}b)^2(a - i\sqrt{2}b)^2$$

which is equivalent to the system of two equations

$$(p - q + i\sqrt{2}q) = (2k - 1 + i\sqrt{2})(a + i\sqrt{2}b)^2$$

$$(p - q - i\sqrt{2}q) = (2k - 1 - i\sqrt{2})(a - i\sqrt{2}b)^2$$

Equating the real and imaginary parts in either of the above two equations and performing an algebra,

we have
$$\begin{aligned} p &= 2k(a^2 - 2b^2) + (4k - 6)ab \\ q &= (a^2 - 2b^2) + (2k - 1)2ab \end{aligned} \quad (5)$$

The condition $p > q > 0$ gives $(2k-1)(a^2 - 3b^2) > 4ab$ (6)

For simplicity and clear understanding, we search for Pythagorean triangles satisfying the relation (2) when k=2, 3

**Case (1):**

Let k=2, Then (3) becomes $(p-q)^2 + 2q^2 = 11\alpha^2$ (7)

From (5) and (6), we have

$$\begin{aligned} p &= 4a^2 - 8b^2 + 2ab \\ q &= a^2 - 2b^2 + 6ab \\ \text{and } 3a^2 &> 6b^2 + 4ab \end{aligned} \quad (8)$$

The above inequality is satisfied when $b = m, a = n + 2m, m, n = 1,2,3,\ldots$

Therefore,
$$\begin{aligned} p &= p(m,n) = 12m^2 + 18mn + 4n^2 \\ q &= q(m,n) = 14m^2 + 10mn + n^2 \end{aligned} \quad (9)$$

In view of (1), the sides of the Pythagorean triangle satisfying (2) are given by

$$x(m,n) = 336m^4 + 744m^3n + 496m^2n^2 + 116mn^3 + 8n^4$$
$$y(m,n) = -52m^4 + 152m^3n + 292m^2n^2 + 124mn^3 + 15n^4$$
$$z(m,n) = 340m^4 + 712m^3n + 548m^2n^2 + 164mn^3 + 17n^4$$

A few numerical examples are presented below

| m | n | p | q | z | $\frac{2A}{P}$ | $z - \frac{4A}{P}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 34 | 25 | 1781 | 225 | $11(11)^2$ |
| 1 | 2 | 64 | 38 | 5540 | 988 | $11(18)^2$ |
| 2 | 1 | 88 | 77 | 13673 | 847 | $11(33)^2$ |
| 2 | 2 | 136 | 100 | 28496 | 3600 | $11(44)^2$ |

**Observation: 1**

From the generators p, q given by (9), one can construct Diophantine quadruple with suitable property.

**Illustration:**

Let $P = p(1,n) - 2n - 12, Q = q(1,n) - 14$ . The quadruple (P,Q,R,S) is a Diophantine quadruple with property $D(36n^2)$ where $R = 9n^2 + 54n, S = 4n^4 + 80n^3 + 524n^3 + 1120n$

**Proof:**

$$PQ + 36n^2 = [2n^2 + 14n]^2$$
$$PR + 36n^2 = [6n^2 + 30n]^2$$
$$PS + 36n^2 = [2n(2n^2 + 24n + 67)]^2$$
$$QR + 36n^2 = [3n^2 + 24]^2$$
$$QS + 36n^2 = [2n(n^2 + 15n + 53)]^2$$
$$RS + 36n^2 = [6n(n^2 + 13n + 41)]^2$$

Thus the quadruple (P,Q,R,S) is a Diophantine quadruple with property $D(36n^2)$

**Note: 1**

(7) is written as $(p-q)^2 = 11\alpha^2 - 2q^2$ (10)

Introduction of the linear transformations

$$\alpha = X + 2T, q = X + 11T \quad (11)$$

in (10) leads to $(p-q)^2 = 9X^2 - 198T^2$

Replacing $(p-q)$ by $3(P-Q)$ in the above equation, it gives

$$(P-Q)^2 + 22T^2 = X^2$$

which is satisfied by

$$T = 2rs, P-Q = 22r^2 - s^2, X = 22r^2 + s^2$$

Substituting the values of X and T in (11), we have

$$\alpha = 22r^2 + 4rs + s^2$$
$$q = 22r^2 + 22rs + s^2 \quad (12)$$

Also, $(p-q) = 3(P-Q) = 66r^2 - 3s^2 2$

And therefore $p = 88r^2 + 22rs - 2s^2$ (13)

Substituting the values of the generators p, q in (1), the sides of the Pythagorean triangle $T(x,y,z)$ satisfying (2) are given by

$$x(r,s) = 3872r^4 + 4840r^3s + 1056r^2s^2 - 44rs^3 - 4s^4$$
$$y(r,s) = 7260r^4 + 2904r^3s - 396r^2s^2 - 132rs^3 + 3s^4$$
$$z(r,s) = 8228r^4 + 4840r^3s + 660r^2s^2 - 44rs^3 + 5s^4$$

A few numerical examples are as follows

| r | s | p | q | z | $\frac{2A}{P}$ | $z - \frac{4A}{P}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 108 | 45 | 13689 | 2835 | $11[27]^2$ |
| 1 | 2 | 124 | 70 | 20276 | 3780 | $11[34]^2$ |
| 2 | 1 | 394 | 133 | 172925 | 34713 | $11[97]^2$ |
| 2 | 2 | 432 | 180 | 219024 | 45360 | $11[108]^2$ |

**Case (2):**

Let k=3, Then (3) becomes $(p-q)^2 + 2q^2 = 27\alpha^2$ (14)

From (5) and (6), we have

$$p = 6a^2 - 12b^2 + 6ab$$
$$q = a^2 - 2b^2 + 10ab \quad (15)$$
$$\text{and } 5a^2 > 10b^2 + 4ab$$

The above inequality is satisfied when $b = m, a = n + 2m - 1$

Therefore,
$$p = p(m,n) = 24m^2 + 6n^2 + 30mn - 30m - 12n + 6$$
$$q = q(m,n) = 22m^2 + n^2 + 14mn - 14m - 2n + 1 \quad (16)$$

In view of (1), the sides of the Pythagorean triangle satisfying (2) are given by

$$x(\mathrm{m,n}) = 1056\mathrm{m}^4 + 1992m^3n + 1152m^2n^2 + 228\mathrm{mn}^3 + 12n^4$$
$$-1992m^3 - 2304m^2n - 684mn^2 - 48n^3 + 1152m^2 + 684mn + 72n^2 - 228m - 48n + 12$$
$$y(\mathrm{m,n}) = 92\mathrm{m}^4 + 824m^3n + 948m^2n^2 + 332\mathrm{mn}^3 + 35n^4$$
$$-824m^3 - 1896m^2n - 996mn^2 - 140n^3 + 948m^2 + 996mn - 210n^2 - 332m - 140n + 35$$
$$z(\mathrm{m,n}) = 1060\mathrm{m}^4 + 2056m^3n + 1428m^2n^2 + 388\mathrm{mn}^3 + 37n^4$$
$$-2056m^3 - 2856m^2n - 1164mn^2 - 148n^3 + 1428m^2 + 1164mn + 222n^2 - 388m - 148n + 37$$

A few numerical examples are presented below

| m | n | p | q | z | $\frac{2A}{P}$ | $z - \frac{4A}{P}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 24 | 22 | 1060 | 44 | $27[6]^2$ |
| 2 | 1 | 96 | 88 | 16960 | 704 | $27[24]^2$ |
| 1 | 2 | 60 | 37 | 4969 | 851 | $27[11]^2$ |
| 1 | 2 | 162 | 117 | 39933 | 5256 | $27[33]^2$ |

**Observation: 2**

From the generators p, q given by (15), one can construct Diophantine triples with suitable property.

**Illustration:**

Let $P = \frac{p(m,2)}{6}, Q = q(m,2)$, the triple (P,Q,R), (Q,R,S) are Diophantine triple with property $D(-7m^4 - 4m^3 + 3m^2 - m)$ where

$$R = 2q(m,2) + 9m + 2 \text{ and } S = 3R(m,2) - 4m^2 - 14m - 3$$

$$PQ + D(-7m^4 - 4m^3 + 3m^2 - m) = [9m^2 + 9m + 1]^2$$

**Proof:**

$$PR + D(-7m^4 - 4m^3 + 3m^2 - m) = [13m^2 + 14m + 2]^2$$
$$QR + D(-7m^4 - 4m^3 + 3m^2 - m) = [31m^2 + 23m + 2]^2$$

Thus, the triple (P,Q,R) is a Diophantine triple with property $D(-7m^4 - 4m^3 + 3m^2 - m)$

Also,

$$QR + D(-7m^4 - 4m^3 + 3m^2 - m) = [31m^2 + 23m + 2]^2$$
$$QS + D(-7m^4 - 4m^3 + 3m^2 - m) = [53m^2 + 37m + 3]^2$$
$$RS + D(-7m^4 - 4m^3 + 3m^2 - m) = [75m^2 + 60m + 6]^2$$

Thus, the triple (Q,R,S) is a Diophantine triple with property $D(-7m^4 - 4m^3 + 3m^2 - m)$

**Note: 2**

(14) is written as $(p-q)^2 = 27\alpha^2 - 2q^2$ (16)

Introduction of the linear transformations

$$\alpha = X + 2T, q = X + 27T \quad (17)$$

in (16) leads to $(p-q)^2 = 25X^2 - 1350T^2$

Replacing $(p-q)$ by $5(P-Q)$ in the above equation, it gives

$$(P-Q)^2 + 54T^2 = X^2$$

which is satisfied by

$$T = 2rs, P - Q = 54r^2 - s^2, X = 54r^2 + s^2$$

Substituting the values of X and T in (17), we have

$$\alpha = 54r^2 + 4rs + s^2$$
$$q = 54r^2 + 54rs + s^2 \quad (18)$$

Also, $(p - q) = 5(P - Q) = 270r^2 - 5s^2$

And therefore $p = 324r^2 + 54rs - 4s^2$ (19)

Substituting the values of the generators p, q in (1), the sides of the Pythagorean triangle $T(x, y, z)$ satisfying (2) are given by

$$x(r,s) = 34992r^4 + 40824r^3s + 6048r^2s^2 - 162rs^3 - 4s^4$$
$$y(r,s) = 102060r^4 + 29160r^3s - 2700r^2s^2 - 540rs^3 + 15s^4$$
$$z(r,s) = 107892r^4 + 40824r^3s + 3348r^2s^2 - 324rs^3 + 17s^4$$

A few numerical examples are presented below

| r | s | p | q | z | $\frac{2A}{P}$ | $z - \frac{4A}{P}$ |
|---|---|---|---|---|---|---|
| 1 | 1 | 374 | 109 | 151757 | 28885 | $27[59]^2$ |
| 1 | 2 | 416 | 166 | 200612 | 41500 | $27[66]^2$ |
| 2 | 1 | 1400 | 325 | 2065625 | 349375 | $27[225]^2$ |

**Conclusion:**

In this paper, infinity many Pythagorean triangle each satisfying the relation Hypotaneous- $4\left(\frac{\text{Area}}{\text{Perimeter}}\right) = (4k^2 - 4k + 3)\alpha^2, k = 2,3$ are obtained. To conclude, one may search for Pythagorean triangle each satisfying the relations consisting of polygonal and pyramidal numbers with suitable ranks.

**References:**

1) L. E. Dickson, “History of theory of numbers”, Volume 2, Chelsea Publishing Company, New York, 1952.
2) D. E. Smith, “History of Mathematics”, Volume1 and 2, Dover Publications, New York, 1953.
3) S. G. Telang, “Number Theory”, Tata McGraw-Hill Publishing Company, New Delhi, 1996.
4) Thomas Koshy, “Elementary Number Theory with Applications”, Academic Press, 2005.
5) T. Nagell, “Introduction to Number Theory”, Plencem, New York, 1988.
6) L. J. Mordell, “Diophantine Equations”, Academic Press, New York, 1969.
7) M. A. Gopalan and S. Leelavathi, “Pythagorean triangle with 2(Area/Perimeter) as a cubic integer”, *Bulletin of Pure and Applied Sciences*, vol.27 E(2), 2007, pp. 197-200.
8) M. A. Gopalan and G. Janaki, “Pythagorean triangle with Area/Perimeter as a special polygonal number”, *Bulletin of Pure and Applied Sciences*, vol. 27 E(2), 2008, pp. 393-402.
9) M.A. Gopalan and S. Devibala, “Pythagorean triangle with triangular number as a leg”, *Impact J.Sci.Tech*. vol.2 (4), 2008, pp.195-199.
10) M.A. Gopalan and G. Janaki, “Pythagorean triangle with nasty number as a leg”, *Journal of Applied Mathematical Analysis and Applications*”, vol.4 (1-2), 2008, pp.13-17.
11) M.A. Gopalan and G. Janaki, “Pythagorean triangle with perimeter as a pentagonal number”, *Antarctica J.Math.*, vol.5 (2), 2008, pp.15-18.

12) M. A. Gopalan and A. Gnanam, "Pythagorean triangles and special polygonal numbers", *International Journal of Mathematical Sciences* vol.9 (1-2), 2010, pp.211-215.
13) M. A. Gopalan and G. Sangeetha, "Pythagoream triangles with perimeter as triangular number", *The Global Journal of Applied Mathematics and Mathematical Sciences*, vol.3 (1-2), 2010, pp.93-97.
14) M. A. Gopalan and B. Sivakami, "Pythagorean Triangle with Hypotenuse minus 2(Area/Perimeter) as a square integer", *Archimedes J.Math.*, vol.2(2), 2012, pp.153-166.
15) M.A. Gopalan and V. Geetha, "Pythagorean triangle with Area/Perimeter as a special polygonal number", *International Refereed Journal of Engineering and Science*, vol.2 (7), 2013, pp.28-34.
16) M. A. Gopalan, Manju Somanath and K. Geetha,"Pythagorean triangle with Area/Perimeter as a special polygonal number", *IOSR-JM*, vol.7 (3), 2013, pp52-62.
17) M. A. Gopalan, Manju Somanath and V.Sangeetha, "Pythagorean triangles and pentagonal number", *Cayley J.Math,*, vol.2(2), 2013, pp.151-156.
18) M. A. Gopalan, Manju Somanath and V. Sangeetha, "Pythagorean triangles and special pyramidal numbers", *IOSR-JM*, vol.7 (4), 2013, pp.21-22.
19) P. Thirunavukkarasu and S. Sriram, "Pythagorean triangle with Area/Perimeter as quartic integer", *International Journal of Engineering and Innovative Technology (IJEIT)*, vol.3 (7), 2014, pp.100-102.
20) M. A. Gopalan, Manju Somanath and V. Sangeetha, "Pythagorean Triangle with Area/Perimeter as a Quartic Integer", *IJERM*, vol-02, Issue-03, March-2015
21) M. A.Gopalan, S. Vidhyalakshmi, N. Thiruniraiselvi, "Pythagorean Triangle with Hypotenuse $-4\left(\frac{Area}{Perimeter}\right)$ is 3 Times a Square Integer' *IJISME*, vol-3, Issue-6, May 2015