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SQUARE – PENTAGONAL NUMBERS

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Abstract:

Among several types of numbers that exists, Figurate numbers play a significant role in understanding the behavior of positive integers. Figurate numbers were studied from ancient times to modern times and they continue to fascinate forever. Among several types of Figurate numbers, the square numbers and pentagonal numbers were very useful sequences of numbers applied in various combinatorial problems. In this paper, I will describe a novel method using continued fraction to determine numbers which are square numbers as well as pentagonal numbers.

Key Words: Figurate Numbers, Square Numbers, Pentagonal Numbers, Continued Fraction, Convergents

1. Introduction:

The study of amusing family of numbers known as Figurate numbers has been done for more than two thousand years. In this paper, I will introduce the *n*th Figurate number of order *k* where $k \geq 3$. For the choices of *k* as 4, 5 we have square and pentagonal numbers respectively. In this paper, I will determine sequence of numbers which are both square as well as pentagonal numbers.

2. Definition:

2.1 The *n*th Figurate number of order *k* is given by the expression

$$P_{k,n} = \frac{n}{2}\left[(k-2)n-(k-4)\right] \quad (1)$$

If $k = 4$ then we get $P_{4,n} = n^2 \ (2)$ and if $k = 5$ then we get $P_{5,n} = \frac{n}{2}(3n-1) = \frac{3n^2-n}{2}$ (3)

In this paper I would determine numbers which are both of forms given by (2) and (3).

3. Determining Square – Pentagonal Numbers:

Using (2), (3), we need to find positive integers *n*, *m* such that $P_{5,n} = P_{4,m}$ (4)

That is, $\frac{3n^2-n}{2} = m^2$ (5). Now multiplying both sides of (5) by 24 and adding 1 to both sides and simplifying we get $(6n-1)^2 - 24m^2 = 1$ (6). If we now assume $x = 6n-1$ then (6) can be re-written as

$$x^2 - 24m^2 = 1 \ (7)$$

Thus equation (5) has now been reduced to a nice Pell's equation as given in (7). To determine solutions of (7), I will derive the following continued fraction.

First, we observe that $(5-\sqrt{24})\times(5+\sqrt{24}) = 1$. Using this, we have the following computations

$$5-\sqrt{24} = \frac{1}{5+\sqrt{24}} = \frac{1}{10-(5-\sqrt{24})} = \frac{1}{10-\cfrac{1}{10-(5-\sqrt{24})}} = \cdots = \cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cdots}}}}}$$

Hence we get

$$\sqrt{24} = 5 - \cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cdots}}}}} \qquad (8)$$

The successive convergents of (8) are given by

$$\frac{5}{1}, 5-\frac{1}{10}=\frac{49}{10}, 5-\cfrac{1}{10-\cfrac{1}{10}}=\frac{485}{99}, 5-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10}}}=\frac{4801}{980}, 5-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10-\cfrac{1}{10}}}}=\frac{47525}{9701},$$

$$5-\cfrac{1}{10-\cfrac{980}{9701}}=5-\frac{9701}{96030}=\frac{470449}{96030}, 5-\cfrac{1}{10-\cfrac{9701}{96030}}=5-\frac{96030}{950599}=\frac{4656965}{950599},$$

$$5-\cfrac{1}{10-\cfrac{96030}{950599}}=5-\frac{950599}{9409960}=\frac{46099201}{9409960}, 5-\cfrac{1}{10-\cfrac{950599}{9409960}}=5-\frac{9409960}{93149001}=\frac{456335045}{93149001},\ldots$$

Considering the numerators and denominators of above rational number convergents as x and m respectively we notice that the pairs $(x,m)=(5,1);(49,10);(485,99);(4801,980);(47525,9701);$ $(470449,96030);(4656965,950599);(46099201,9409960);(456335045,93149001);\ldots$ form solutions to $x^2-24m^2=1$.

Since $n=\frac{x+1}{6}$ from the above solutions, we notice that for second, fourth, sixth, eighth pairs listed above, n is not an integer. Hence, we can neglect such pairs and consider rest to get the required solution. Hence using the pairs

$$(x,m)=(5,1);(485,99);(47525,9701);(4656965,950599);(456335045,93149001),\ldots$$

We find that

$$(n,m)=(1,1);(81,99);(7921,9701);(776161,950599);(76055841,93149001),\ldots$$

are solutions of the equation (5) given by $\frac{3n^2-n}{2}=m^2$.

Thus the first few Square – Pentagonal numbers are given by 1, 9801, 94109401, 903638458801, 8676736387298001 . . .

By extracting more alternate convergents from the continued fraction (8), we can obtain infinitely many Square – Pentagonal numbers.

4. Conclusion:

The main concept of this paper is to determine the equality of two types of Figurate numbers namely square numbers and pentagonal numbers. Using the continued fraction obtained in expression (8), I had obtained Square – Pentagonal numbers in this paper. This method is very elegant and compact compared to other known methods to obtain such numbers. One can similarly construct appropriate continued fraction expansion to determine solutions for Square – Triangular numbers, Square – Hexagonal numbers and so forth. This paper provides a template for doing so.

5. References:

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